PERMUTATION SYMMETRY

IN

WEAK INTERACTIONS

A THESIS

Presented to

The Faculty of the Graduate Division

by

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in the School of Physics

Georgia Institute of Technology

August, 1966

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Approved:

Chairman

Date approved by Chairman July 15, 1966

ACKNOWLEDGMENTS

The author wishes to thank Dr. R. M. Ahrens for many discussions from which the basic ideas of this thesis evolved and for his supervision of the investigation. The author wishes also to thank Dr. C. H. Braden for many enlightening discussions on diverse topics and Dr. H. R. Brewer for several comments on raising and lowering operators and their relation to Clebsch-Gordan coefficients.

A special note of appreciation is due to the author's family whose faithful support never diminished during the many years that the author spent as a student.

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SUMMARY

The weak interaction is responsible for those decays of the unstable elementary particles which are forbidden by the stronger interactions. It is also responsible for certain scattering processes. If strange particles are not included the weak interaction appears to be well described, at least to first order in the weak coupling constant, by a universal V-A parity violating interaction. In order to include the interactions of strange particles it appears to be necessary to go outside the scope of the universal V-A theory. The Cabibbo theory of leptonic weak interactions is a rather successful attempt in this direction. It is based on the rather well established unitary symmetry of the strongly interacting particles. In the Cabibbo theory the relative strengths of strangeness changing and strangeness conserving processes are determined in terms of the Cabibbo angle which must be found empirically. It is hoped that in a more complete theory these relative strengths will be determined from "first principles." The objective of this thesis is to obtain a self-contained picture of weak interactions which predicts the occurrence of certain processes along with their relative coupling constants.

Now it is known that the pure V-A interaction in the absence of strong interaction renormalization effects is P_{13} invariant, i.e., it is invariant under the interchange of the first and third Dirac 4-spinors which make up the interaction. The significance of this invariance property is not known and, in fact, it is usually ignored. In this

investigation, however, the P_{13} invariance of the V-A interaction is given special significance. In fact, by generalizing this invariance to the internal coordinate space of the elementary particles the above objective is accomplished.

The investigation is carried out in four steps:

1. A reordering theorem analogous to the Fierz reordering theorem is proved for an arbitrary semisimple compact group.
2. The reordering theorem is applied to the SU(3) symmetry scheme of Gell-Mann and Ne'eman for strongly interacting particles in order to determine the P_{13} invariant SU(3) scalars.
3. A particular model is chosen for the leptons and baryons within the Gell-Mann-Ne'eman scheme.
4. The consequences of the model are compared with the universal V-A theory, with the Cabibbo theory and with experimental results.

The reordering theorem proved in this thesis states that if the first and third factors in any one of the biquadiatic scalars associated with an irreducible representation of a simisimple compact group are interchanged the result can be expressed as a linear combination of the biquadratic scalars with the original order. A similar theorem, the Fierz reordering theorem, which applies to biquadratic Lorentz scalars has been known for some time. Furthermore K. M. Case has shown that a Fierz type reordering theorem exists for all complex orthogonal groups. However, these are not in general semisimple compact groups so that Case's result does not apply to the groups discussed in this thesis.

The proof of the reordering theorem given in this thesis requires that the Clebsch-Gordan coefficients for the group be real. Although it is known in the literature that the Clebsch-Gordan coefficients for a compact group can be selected in such a way as to form a unitary matrix, it is apparently not known that they can always be made real. This fact is proved as a preliminary to the proof of the reordering theorem.

In order to carry out the second step of the investigation, the reordering matrix, which is the matrix of coefficients in the reordering theorem, for the Gell-Mann-Ne'eman octet model is evaluated along with an 8×8 matrix which diagonalizes the reordering matrix. From this latter matrix the P_{13} invariants can be written down immediately in terms of the biquadratic scalars associated with the octet model. It is also shown that in the case of SU(3) the elements of the reordering matrix are simply related to the generalized Racah W-coefficients for that group. In addition a complete discussion as well as the definition of the generalized Racah W-coefficients for an arbitrary semisimple compact group is given.

The particular model chosen for the leptonic decays of the baryons is based on the following assumptions:

1. As far as the baryons are concerned the weak interaction is a P_{13} invariant SU(3) biquadratic scalar with eigenvalue -1. The -1 eigenvalue is chosen since a similar analysis in the SU(2) isospin space results in a P_{13} invariant which is physically reasonable and which has eigenvalue -1.
2. The lepton current can replace any quadratic factor in the P_{13} invariant baryon expression which has commutation relations similar to those of the lepton current and its hermitian conjugate. The full significance of this assumption is made clear

by explicit calculation.

3. The weak interaction is as simple as possible under assumptions 1 and 2. This rather aesthetic assumption is perhaps unjustified except that in a certain sense it makes the model unique.

Under these assumptions a unique interaction is written down for the leptonic decays of the baryons. This interaction has the remarkable property that it involves only the eight-dimensional representation of $SU(3)$. Furthermore it is the only P_{13} invariant biquadratic $SU(3)$ scalar which has this property.

In order to compare this model with previous models and with experimental results, the branching ratios for various processes are evaluated using a formula derived in this thesis which is valid to fourth order in the mass difference between the initial and final baryons. The results indicate that this model is at least as good as the universal V-A theory and in several cases is a significant improvement over the universal V-A theory. However the results do not match the agreement achieved by the empirically adjusted Cabibbo theory.

CHAPTER I

INTRODUCTION

Historical Development of Weak Interactions

It is reasonable to say that weak interaction physics was born with the discovery of the electron in 1897 by J. J. Thomson (1) and the isolation of beta rays in nuclear radioactivity by Rutherford (2) in 1899. Toward the end of 1899 the nuclear beta rays were identified with Thomson's electron.

No significant progress in the understanding of the nuclear beta decay process was made until after the formulation of the Dirac equation (3) describing relativistic spin 1/2 particles in 1928 and the discovery of the neutron by Chadwick (4) in 1932. In the year previous to the discovery of the neutron Pauli (5) had proposed the existence of a massless, chargeless particle, the neutrino, in order to account for the observed continuous energy spectrum in beta decay. These developments paved the way for Fermi's (6) monumental work on the theory of beta decay. Fermi based his theory on an analogy with the vector current interaction of quantum electrodynamics which had been developing in the previous eight years. In his theory a Dirac vector covariant involving the electron and neutrino played the role of the electromagnetic vector potential while the part of the electromagnetic current for charged particles was played by another Dirac vector covariant involving the nuclear protons and neutrons. Inherent in Fermi's theory is the assumption that beta decay occurs by virtue of a four fermion point interaction and that the electron is formed at the moment of interaction. This avoids the difficulty of confining a light particle like an electron to orbits of nuclear dimensions.

In the allowed approximation (i.e., non-relativistic nucleons and low momentum transfer) the Fermi theory forbids decays in which the nuclear spin changes. Gamow and Teller (7) introduced a Dirac axial vector interaction in 1936 in order to permit beta decays in the allowed approximation for which the nuclear spin changed by one unit. Although these arguments were in fairly good agreement with the experimental results on lifetimes and spectrum shapes, the appropriate combination of the vector and axial vector interactions was not known. Indeed, it was not conclusive that the vector and axial vector interactions were the appropriate choices since there are also scalar and tensor interactions which have the same selection rules, respectively, as the vector and axial vector interactions. The correct form of the beta decay interaction was arrived at only after twenty years of meticulous experimental research guided by theoretical considerations. Most important among the theoretical contributions was the suggestion by Lee and Yang (8), in 1956, that parity might be violated in weak interactions. Parity violation in beta decay was experimentally confirmed to be a maximum by Wu and her collaborators (9) in 1957. The following year the vector-axial vector interaction was chosen over the scalar-tensor interaction (10). Two years later the proportion of axial vector to vector contribution was established by electron-neutrino angular correlation measurements (11). Thus by the late 1950's it could be safely said that the beta decay interaction was, at least phenomenologically, V-xA with maximum parity violation where $-x$ is the proportion of axial vector (A) to vector (V) interaction.

Further interest in weak interactions had been stimulated by the discovery of the muon (12) in cosmic rays in 1936, the year of the Gamow-Teller paper, and the discovery in cloud chamber photographs of the strange particles (13) in 1947. An immediate difficulty arose with the discovery of the new strange particles. These particles were copiously produced, indicating a strong interaction; however, they decayed slowly, indicating a weak interaction. To explain this, Gell-Mann, and independently Nishijima (14), introduced, in 1953, a new quantum number, strangeness, which was presumed to be conserved in strong interactions but not conserved in weak interactions.

The muon was originally thought to be Yukawa's meson, hypothesized earlier as the mediator of nuclear interactions (15) but it soon became apparent that the muon interacted far too weakly with nuclear matter to be the Yukawa particle. The true Yukawa particle, the pion, was discovered (16) the same year that the strange particles were first observed.

The pions and muons had decay properties characteristic of the nuclear beta decays. Hence it was felt as early as 1947 that these decays, along with orbital electron capture, were somehow explainable by a single interaction (17), a sort of universal Fermi interaction (U.F.I.). In fact, in the course of time it was established that the muon decay was describable by a pure V-A interaction with maximum parity violation and having apparently the same vector coupling constant as in the beta decay interaction.

Although the ratio x of axial vector to vector interaction in beta dacay was not one, it was sufficiently close to one ($x \approx 1.18$) to encourage physicists to blame the discrepancy on renormalization effects due to strong interactions. Such renormalization effects, of course, would not appear in the muon decay since none of the particles involved interact strongly. This explanation of the difference of x from one immediately raises the question: Why, if the axial vector coupling constant is renormalized by strong interaction effects, is not the vector coupling constant likewise renormalized? Feynman and Gell-Mann (18) in 1958 proposed the following solution to this question. They assumed that the weak interaction could be written as a product of two currents and that the vector part of the hadron (strongly interacting particle) current was conserved, i.e., it was assumed to have zero divergence. This implied that the weak interaction vector coupling constant suffered no apparent renormalization effects when hadrons were involved.

A similar result occurs in electrodynamics where the electromagnetic coupling constant (charge) is unrenormalized because of the conserved electromagnetic current.

This explanation of the absence of renormalization of the vector coupling constant has become known as the conserved vector current theory (CVCT). However, this name usually connotes more than the mere assumption of the conserved vector current. It is also assumed in the CVCT that the weak vector current belongs to the same isospin multiplet as the electromagnetic current so that simple relations exist between similar weak and electromagnetic processes. These relations have been substantially born out by experiment (see ref. 19, p. 402).

Hence it seemed reasonable to assume that the weak interactions of non-strange particles were indeed governed by a universal interaction which, aside from strong interaction renormalization effects, was pure V-A with maximum parity violation.

Dallaporta (20) suggested in 1955 that even the strange particle decays might be governed by the same single interaction. However, strange particle lifetime estimates based on this assumption were at least an order of magnitude too small. In addition to this difficulty very careful measurements of the vector coupling coefficients in beta decay and muon decay as late as 1962 showed that these two coupling constants were in fact different by about 2% (see ref. 21, p.5). This discrepancy was essentially removed and the problem of the strange particle lifetimes was partially alleviated when Cabibbo (22) abandoned the U.F.I. in favor of a theory based on the Gell-Mann-Ne'eman (23) SU(3) symmetry scheme for hadrons. A complete discussion of the Cabibbo theory appears in a later section of this chapter.

The Eightfold Way

The Gell-Mann-Ne'eman scheme, commonly called the eightfold way or octet model, assumes that it's legitimate to think of the strong interaction, which acts among baryons and mesons, as consisting of two parts: a part which is invariant under the special unitary group in three dimensions, SU(3), and a symmetry breaking part which is not invariant under SU(3). This means that in the limit of unitary (i.e., SU(3)) symmetry the mesons and baryons fit into supermultiplets which transform according to irreducible representations of SU(3). Such supermultiplets must consist of one or more isospin multiplets since isospin is known to be conserved by the total strong interaction. In order to determine these supermultiplets it will be necessary to look at the formalism of SU(3) .

First of all SU(3) is the set of all 3 x 3 unitary unimodular (i.e., determinant plus one) matrices. Since these matrices are 3 x 3 and complex each one involves 18 real numbers. However only eight of these are independent because of the unitary and unimodular conditions. This is easily seen by noting that every matrix of SU(3) can be written in the form

$$u = e^{ih}$$

where h is hermitian (i.e., u is unitary) and traceless (i.e., u is unimodular). Furthermore the elements of the matrices can be written as continuous functions of these eight independent parameters. Hence a great deal of information can be obtained by looking at the infinitesimal elements of the group. In particular if h is infinitesimal it can be expressed to a first approximation as a linear function of the eight independent parameters. Thus

$$h = \frac{1}{2} \sum_{k=1}^{8} \varepsilon_k \lambda_k$$

where the ε_k are the eight real parameters, the λ_k are eight linearly independent 3 x 3 hermitian, traceless matrices and the factor of 1/2 is inserted for convenience. The usual choices for the λ_k , those of Gell-Mann (23), are shown in Table 1. It should be noted that the first three, λ_1 , λ_2 , and λ_3 , are analogous to the usual isospin matrices. In

Table 1. The λ_k

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

addition these matrices satisfy the following relations*:

$$\mathrm{Tr}\, \lambda_i \lambda_j = 2\, \delta_{ij}$$

$$\left[\lambda_i, \lambda_j\right] = 2\, i\, f_{ijk}$$

$$\left\{\lambda_i, \lambda_j\right\} = \frac{4}{3}\, \delta_{ij} + 2\, d_{ijk}\, \lambda_k$$

where f_{ijk} is real and totally antisymmetric while d_{ijk} is real and totally symmetric. Also $\left[\lambda_i, \lambda_j\right]$ denotes the commutator of λ_i and λ_j while $\left\{\lambda_i, \lambda_j\right\}$ denotes the anticommutator. The f_{ijk} and d_{ijk} are listed in Table 2.

*This discussion follows that of Gell-Mann's original paper.

Table 2. Non-zero Elements of f_{ijk} and d_{ijk}

ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	-1/2	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	-1/2
345	1/2	256	1/2
367	-1/2	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	-1/2
		377	-1/2
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

Note: The f_{ijk} are odd under interchange of any two indices while the d_{ijk} are even.

If ℓ is a column matrix of three quantities which transform according to the self representation of SU(3), then under an infinitesimal transformation of SU(3)

$$\ell \longrightarrow (1 + i \tfrac{1}{2} \varepsilon_k \lambda_k)\ell$$

Furthermore if $\bar{L}$ denotes the hermitian conjugate of a similar set of quantities, then

$$\bar{L} \longrightarrow \bar{L}(1 - i \tfrac{1}{2} \varepsilon_k \lambda_k)$$

Now from the nine possible products of elements of $\bar{L}$ and ℓ one can form the following linear combinations.

(1) $\bar{L}\ell$

This transforms as a scalar under SU(3). This follows immediately from the transformation properties of $\bar{L}$ and ℓ.

$$\bar{L}\ell \longrightarrow \bar{L}(1 - i \tfrac{1}{2} \varepsilon_k \lambda_k)(1 + i \tfrac{1}{2} \varepsilon_j \lambda_j)\ell = \bar{L}\ell + O(\varepsilon^2)$$

where $O(\varepsilon^2)$ denotes terms which are second order in ε_k. Thus to first order $\bar{L}\ell \longrightarrow \bar{L}\ell$.

(2) $\bar{L}\lambda_i\ell$

These form a closed set under SU(3). Again this follows from the transformation properties of $\bar{L}$ and ℓ as well as the commutation relations of the λ_k.

$$\bar{L}\lambda_i\ell \longrightarrow \bar{L}(1 - i \tfrac{1}{2} \varepsilon_k \lambda_k)\lambda_i(1 + i \tfrac{1}{2} \varepsilon_j \lambda_j)\ell$$

$$= \bar{L}\lambda_i\ell + i \tfrac{1}{2} \varepsilon_j \bar{L}[\lambda_i, \lambda_j]\ell + O(\varepsilon^2)$$

$$= \bar{L}\lambda_i\ell - \varepsilon_j f_{ijk} \bar{L}\lambda_k\ell + O(\varepsilon^2)$$

Thus to first order in ε

$$\bar{L}\lambda_i \ell \longrightarrow (\delta_{ij} + \varepsilon_j f_{jik})\bar{L}\lambda_k \ell$$

where the antisymmetry of the f_{ijk} has been used.

By introducing the 8 x 8 matrices F_i defined by

$$(F_i)_{jk} = -if_{ijk}$$

this expression becomes

$$\bar{L}\lambda_i \ell \longrightarrow (\delta_{ik} + i\,\varepsilon_j (F_j)_{ik})\bar{L}\lambda_k \ell$$

or

$$N \longrightarrow (1 + i\,\varepsilon_j F_j)N$$

where N is a column matrix consisting of the eight $\bar{L}\lambda_i \ell$. Now by making use of the Jacobi identity

$$[[\lambda_i, \lambda_j], \lambda_k] + [[\lambda_j, \lambda_k], \lambda_i] + [[\lambda_k, \lambda_i], \lambda_j] = 0$$

it follows that

$$[F_i, F_j] = i\, f_{ijk} F_k$$

Thus the F_i form an eight-dimensional representation of the $\frac{1}{2}\lambda_i$. This representation is in fact irreducible. In other words the components of N transform according to an eight dimensional irreducible representation of SU(3) . These results establish the decomposition

$$3^* \times 3 = 1 + 8$$

where the irreducible representations of SU(3) are denoted by their dimensions. The asterisk denotes the complex conjugate representation.

Further analysis establishes the following decomposition which will be used in subsequent discussions

$$8 \times 8 = 1 + 8 + 8 + 10 + 10^* + 27$$

The double occurrence of the eight-dimensional representation in this decomposition deserves special attention. This double occurrence means that there are two ways of combining two sets of quantities like N above in order to obtain quantities which transform according to the eight-dimensional representation of SU(3) . As has already been shown by Gell-Mann, these two ways can be written as $\bar{N} F_i N$ and $\bar{N} D_i N$ where the matrices D_i are defined by $(D_i)_{jk} = d_{ijk}$. Two things should be pointed out here:

(1) 8^* is equivalent to 8;

(2) the elements of F_i and D_i as well as of λ_i in the expression for N_i play the role of generalized Clebsch-Gordan coefficients.

Now one of the principal assumptions of the eightfold way is that the eight $J^P = \frac{1}{2}^+$ baryons form a supermultiplet which transforms according to the eight-dimensional representation of SU(3) (i.e., they form an SU(3) octet or octuplet). The assumed transformation properties of the baryons are given in Table 3. These assignments were made by Gell-Mann on the basis of isospin, electric charge and hypercharge.

Table 3. Transformation Properties of the $\frac{1}{2}^{+}$ Baryons

Baryon	Transforms Like
Σ^{+}	$1/2\ \bar{L}(\lambda_1 - i\ \lambda_2)\ \ell$
Σ^{-}	$1/2\ \bar{L}(\lambda_1 + i\lambda_2)\ \ell$
Σ^{o}	$1/\sqrt{2}\ \bar{L}\ \lambda_3\ \ell$
p	$1/2\ \bar{L}(\lambda_4 - i\ \lambda_5)\ \ell$
n	$1/2\ \bar{L}(\lambda_6 - i\ \lambda_7)\ \ell$
Ξ^{o}	$1/2\ \bar{L}(\lambda_6 + i\ \lambda_7)\ \ell$
Ξ^{-}	$1/2\ \bar{L}(\lambda_4 + i\ \lambda_5)\ \ell$
Λ	$\frac{1}{\sqrt{2}}\ \bar{L}\ \lambda_8\ \ell$

<u>Cabibbo Theory</u>

It is generally assumed that the leptonic decays of the hadrons can be described by an interaction of the form

$$H_L = \frac{G}{\sqrt{2}} J_\mu L_\mu + \text{h.c.}$$

where J_μ is the weak hadron current, L_μ the weak lepton current and h.c. denotes the hermitian conjugate of the first term.

Cabibbo* assumed that J_μ has the form:

$$J_\mu = \cos\Theta \left[j^1_\mu + g^1_\mu + i(j^2_\mu + g^2_\mu) \right] + \sin\Theta \left[j^4_\mu + g^4_\mu + i(j^5_\mu + g^5_\mu) \right]$$

where the j^i_μ are vector currents which transform like $\bar{L}\lambda_i \ell$ under SU(3) and the g^i_μ are axial vector currents which also transform like $\bar{L}\lambda_i \ell$ under SU(3). Thus the $\cos\Theta$ term accounts for strangeness conserving processes while the $\sin\Theta$ term accounts for strangeness changing processes. The angle Θ determines the relative strength of these two types of processes. Cabibbo determined Θ empirically to be about 0.26. Thus the $\cos\Theta$ results in a reduction of the beta decay coupling constant relative to the muon coupling constant by about 3% while the $\sin\Theta$ results in a decrease of the strangeness changing coupling constant by about a factor of four relative to that of the strangeness conserving processes.

In addition to the above form of J_μ Cabibbo assumed that the vector part of J_μ belongs to the same octet as the electromagnetic current J^{em}_μ. In the octet model

$$J^{em}_\mu = j^3_\mu + \frac{1}{\sqrt{3}} j^8_\mu$$

This, in effect, builds the conserved vector current theory into the Cabibbo theory. Thus definite relations exist between the matrix elements of the vector part of J_μ and the matrix elements of J^{em}_μ. The exact relationship can be obtained by using a generalization** of the

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*The following discussion follows that of reference 22.

**See reference 24 for a discussion of this generalization to SU(3).

Wigner-Eckart theorem (25, 26). If A and B belong to an octet of SU(3) , then

$$\langle A|j^i_\mu|B\rangle = i\, f_{ABi}\, O_\mu + d_{ABi}\, E_\mu$$

Here O_μ and E_μ play the role of reduced matrix elements which depend on the spatial structure of the matrix element but do not depend on the particular components of the octet involved. As was pointed out before, the f_{ABi} and d_{ABi} play the role of Clebsch-Gordan coefficients. Two reduced matrix elements appear here since the eight-dimensional representation appears twice in the decomposition of two octets. In the allowed approximation

$$O_\mu = F^O \langle \gamma_\mu \rangle$$

and

$$E_\mu = F^E \langle \gamma_\mu \rangle$$

where the F's are constants and $\langle \gamma_\mu \rangle$ is the usual vector matrix element.

Using Tables 2 and 3 one finds that

$$\langle p|J^{em}_\mu|p\rangle = 1/2(1/3\, F^E + F^O) \langle \gamma_\mu \rangle$$

and

$$\langle n|J^{em}_\mu|n\rangle = -1/3\, F^E \langle \gamma_\mu \rangle$$

Since it is known that in the allowed approximation

$$\langle p | J_\mu^{em} | p \rangle = \langle \gamma_\mu \rangle$$

and

$$\langle n | J_\mu^{em} | n \rangle = 0$$

it follows that

$$F^E = 0 \quad \text{and} \quad F^O = 2$$

Now for the weak processes it is necessary to generalize O_μ and E_μ to include the axial vector contributions. Thus

$$\langle A | j_\mu^i + g_\mu^i | B \rangle = i f_{ABi}\, O_\mu + d_{ABi}\, E_\mu$$

where now in the allowed approximation

$$O_\mu = F^O \langle \gamma_\mu \rangle + H^O \langle \gamma_\mu \gamma_5 \rangle$$

and

$$E_\mu = F^E \langle \gamma_\mu \rangle + H^E \langle \gamma_\mu \gamma_5 \rangle$$

Here the H's are constants and $\langle \gamma_\mu \gamma_5 \rangle$ is the axial vector matrix element. Of course, the vector part is determined by the electromagnetic matrix elements so that

$$F^E = 0 \quad \text{and} \quad F^O = 2$$

thus

$$O_\mu = 2 \langle \gamma_\mu \rangle + H^O \langle \gamma_\mu \gamma_5 \rangle$$

and

$$E_\mu = H^E \langle \gamma_\mu \gamma_5 \rangle$$

Apparently H^O and H^E must be determined empirically.

The matrix elements of the weak hadron current can now be determined for any process in the allowed approximation and in the limit of unitary symmetry. This has been done previously by several authors. Feynman (27) appears to give the most nearly complete list of results. Only a few will be recorded here.

$$\langle p|J_\mu|n\rangle = \cos\Theta\, [\langle \gamma_\mu \rangle + \tfrac{1}{2}(H^E + H^O) \langle \gamma_\mu \gamma_5 \rangle]$$

$$\langle \Lambda|J_\mu|\Sigma^-\rangle = \sqrt{2/3}\, \cos\Theta\, (\tfrac{1}{2} H^E) \langle \gamma_\mu \gamma_5 \rangle$$

$$\langle p|J_\mu|\Lambda\rangle = -\sqrt{3/2}\, \sin\Theta\, [\langle \gamma_\mu \rangle + \tfrac{1}{2}(\tfrac{1}{3} H^E + H^O) \langle \gamma_\mu \gamma_5 \rangle]$$

$$\langle n|J_\mu|\Sigma^-\rangle = \sin\Theta\, [\langle \gamma_\mu \rangle - 1/2(H^E - H^O) \langle \gamma_\mu \gamma_5 \rangle]$$

By using recent experimental data (19, 28) and the lifetime expression to be derived in Chapter V, the first two matrix elements can be used to evaluate H^E and H^O. The results are

$$1/2\, H^E = 0.81 \pm 0.14$$

and

$$1/2\, H^O = 0.37 \pm 0.16$$

The factors of a half have been inserted for easy comparison with other results in the literature. From these values for H^E and H^O one can now evaluate the branching ratios for leptonic decays described by the last two matrix elements above. The results are given in Table 4 along with the experimental results and the U.F.I. results.

Table 4. Branching Ratios $(X10^3)$ for Baryon Beta Decays

Decay	Experiment	U.F.I.	Cabibbo
$\Lambda \rightarrow p e^- \bar{\nu}$	0.85 ± 0.09	19.1 ± 3.1	$0.55 \begin{smallmatrix} + 0.31 \\ - 0.21 \end{smallmatrix}$
$\Sigma^- \rightarrow \Lambda e^- \bar{\nu}$	1.3 ± 0.2	70.6 ± 5.5	$0.99 \begin{smallmatrix} + 0.72 \\ - 0.35 \end{smallmatrix}$
$\Sigma^- \rightarrow \Lambda e^- \bar{\nu}$	0.074 ± 0.02	0.296 ± 0.012	$0.074 \pm 0.02^*$

*input information

It's clear from these results that the Cabibbo theory is in much better agreement with experiment than is the U.F.I. Furthermore Cabibbo's theory has built into it four of the selection rules customarily associated with the weak interactions:

(1) the $\Delta S = 0, \pm 1$ rule, where ΔS is the change in strangeness of the hadrons;

(2) the $\Delta Q = \Delta S$ rule (18) for $\Delta S = \pm 1$ reactions involving leptons, where ΔQ is the change in charge of the hadrons;

(3) the $|\Delta \vec{I}| = \frac{1}{2}$ rule (29) for $\Delta S = \pm 1$ leptonic reactions,

where $\Delta \vec{I}$ is the change in the isopin of the hadrons;

(4) the $|\Delta \vec{I}| = 1$ rule (30) for $\Delta S = 0$ leptonic reactions.

However, in closing this section it should be pointed out that:

(1) the Cabibbo theory does not determine the relative strengths of strangeness changing processes and strangeness conserving processes from "first principles;"

(2) the Cabibbo theory when extended to nonleptonic processes as done, for example, by J. S. Bell (31) leaves an asymmetry between baryons and leptons since it uses neutral hadron currents but not neutral lepton currents.

Additional Contributions

Table 5 summarizes some of the important contributions to weak interaction physics which were not discussed in this brief historical sketch.

Fierz Reordering Theorem

Of the contributions listed in Table 5 the Fierz reordering theorem (35) will be of extreme interest in the following chapters. Thus special consideration will be given to it in this section.

It is well known that by using the Dirac matrices discussed in Appendix 1 five Lorentz scalars can be formed from four 4-spinors. By using the summation convention on repeated indices these biquadratic scalars can be written in the following fashion.

$$S_1 = \bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4$$

$$S_2 = \bar{\psi}_1 \gamma_\mu \psi_2 \bar{\psi}_3 \gamma_\mu \psi_4$$

Table 5. Additional Contributions to the Physics of Weak Interactions

Year	Contribution
1929	Two component neutrino theory (32)
1932	Positron discovered (33)
1932	Isospin formalism introduced (34)
1937	Fierz reordering theorem (35)
1951	Free neutron decay detected (36)
1953	Neutrino detected directly (37)
1954	TCP theorem (38)
1955	K^0 regeneration (39)
1955	Lepton conservation (40)
1957	Two component neutrino theory in beta decay (41)
1957	Most general four fermion interaction (42)
1958	Current-current interaction and resurrection of intermediate boson (18)
1958	Goldberger-Treiman relation (43)
1960	Supposition of neutral hadron currents (44)
1960	Schizon theory (45)
1962	Experimental confirmation of muon neutrino $\neq$ electron neutrino (46)
1963	Beta decay of the pion and weak magnetism (47)
1964	Time reversal noninvariance in K° decay (48)

$$S_3 = \bar{\psi}_1 \sigma_{\mu\nu} \psi_2 \bar{\psi}_3 \sigma_{\mu\nu} \psi_4 , \quad \sigma_{\mu\nu} = \frac{1}{2i} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$$

$$S_4 = \bar{\psi}_1 i \gamma_\mu \gamma_5 \psi_2 \bar{\psi}_3 i \gamma_\mu \gamma_5 \psi_4 , \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$$

$$S_5 = \bar{\psi}_1 \gamma_5 \psi_2 \bar{\psi}_3 \gamma_5 \psi_4$$

Since the formulation of the Dirac equation in 1928 several authors have considered the behavior of these five scalars under permutation of either the "barred" or "unbarred" spinors (49). The interchange of the first and third spinors is referred to as the P_{13} operation while interchange of the second and fourth spinors is called the P_{24} operation. The result of this P_{13} (or P_{24}) operation on the Lorentz scalars is contained in the Fierz reordering theorem. Complete discussions of this theorem are available in several places (see ref. 50). Consequently only the result in matrix notation will be quoted here:

$$P_{13} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{pmatrix} = -1/4 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{pmatrix}$$

The overall minus sign arises from the assumption that the different ψ's anticommute. In the following the matrix of coefficients will be referred to as the Fierz reordering matrix.

It should be noted here that the five scalars S_i form a closed set under the P_{13} operation*. Hence it would not be unreasonable to suspect that certain linear combinations of the biquadratic scalars are invariant up to a sign under P_{13}. This is indeed the case. In fact $S_2 - S_4$ which was previously referred to as the V - A interaction is such an invariant with eigenvalve -1, as can easily be seen by inspection of the Fierz reordering matrix.

The combination $S_2 - S_4$ is, of course, parity conserving. Because of the parity violating nature of weak interactions it is appropriate to consider here the biquadratic Lorentz pseudoscalars. K. M. Case (51) has given the reordering matrix for these pseudoscalars. By inspection of Cases' result and by taking the above discussion into account one can see that the V - A interaction <u>with</u> maximum parity violation is also a P_{13} invariant. Rather than resorting to the formalism used by Case this result can be obtained in a straight forward manner by explicit calculation using the representation of the Dirac matrices in Appendix 1. Because of its informative nature this calculation will be done in some detail.

$$S_2 - S_4 = \bar{\psi}_1 \gamma_\mu \psi_2 \bar{\psi}_3 \gamma_\mu \psi_4 + \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 \bar{\psi}_3 \gamma_\mu \gamma_5 \psi_4$$

This expression can be made maximum parity violating by writing $\frac{1}{\sqrt{2}}(1+\gamma_5)$ in front of each ψ_4. Thus the weak interaction is described by expressions of the form

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*Of course, it is clear that any statements made about P_{13} are equally valid for P_{24}.

$$H_w = \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma_\mu \psi_2 \bar{\psi}_3 \gamma_\mu (1 + \gamma_5)\psi_4 + \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 \bar{\psi}_3 \gamma_\mu \gamma_5 (1+\gamma_5)\psi_4$$

where g is the weak interaction coupling constant. Since $\gamma_5^2 = 1$, some trivial rearrangements result in

$$\begin{aligned} H_w &= \frac{g}{\sqrt{2}} \bar{\psi}_1 \gamma_\mu (1 + \gamma_5) \psi_2 \bar{\psi}_3 \gamma_\mu (1 + \gamma_5)\psi_4 \\ &= \frac{g}{4\sqrt{2}} \bar{\psi}_1^L \gamma_\mu \psi_2^L \bar{\psi}_3^L \gamma_\mu \psi_4^L \end{aligned}$$

where $\psi^L = (1 + \gamma_5)\psi$. The expression for H_w in terms of the ψ^L follows from the anticommutation properties of the Dirac matrices.

$$\gamma_4 \gamma_\mu (1 + \gamma_5) = \frac{1}{2} \gamma_4 \gamma_\mu (1 + \gamma_5)^2 = \frac{1}{2} (1 + \gamma_5)\gamma_4 \gamma_\mu (1 + \gamma_5)$$

Now setting

$$\psi = \begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix}$$

it follows that

$$\psi^L = (1 + \gamma_5)\psi = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix} = \begin{pmatrix} \psi_u - \psi_\ell \\ -\psi_u + \psi_\ell \end{pmatrix} = \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}$$

and

$$\bar{\psi}^L = (\psi^L)^\dagger \gamma_4 = (\varphi^\dagger \; -\varphi^\dagger) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\varphi^\dagger \; \varphi^\dagger)$$

where $\varphi = \psi_u - \psi_\ell$ and $\dagger$ denotes hermitian conjugation. Substituting in H_w gives

$$H_w = \frac{g}{4\sqrt{2}}\left[(\varphi_1^\dagger\,\varphi_1^\dagger)\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix}\begin{pmatrix}\varphi_2\\ -\varphi_2\end{pmatrix}\ (\varphi_3^\dagger\,\varphi_3^\dagger)\begin{pmatrix}0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix}\begin{pmatrix}\varphi_4\\ -\varphi_4\end{pmatrix}\right.$$

$$\left.+\ (\varphi_1^\dagger\,\varphi_1^\dagger)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_2\\ -\varphi_2\end{pmatrix}\ (\varphi_3^\dagger\,\varphi_3^\dagger)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_4\\ -\varphi_4\end{pmatrix}\right]$$

$$= -\varphi_1^\dagger\,\vec{\sigma}\,\varphi_2 \cdot \varphi_3^\dagger\,\vec{\sigma}\,\varphi_4 + \varphi_1^\dagger\,\varphi_2\,\varphi_3^\dagger\,\varphi_4$$

Setting $\varphi = \begin{pmatrix}a\\ b\end{pmatrix}$ and using the usual representation of the Pauli matrices gives

$$\begin{aligned}
H_w = &- (b_1^\dagger a_2 + a_1^\dagger b_2)(b_3^\dagger a_4 + a_3^\dagger b_4)\\
&+ (b_1^\dagger a_2 - a_1^\dagger b_2)(b_3^\dagger a_4 - a_3^\dagger b_4)\\
&- (a_1^\dagger a_2 - b_1^\dagger b_2)(a_3^\dagger a_4 - b_3^\dagger b_4)\\
&+ (a_1^\dagger a_2 + b_1^\dagger b_2)(a_3^\dagger a_4 + b_3^\dagger b_4)\\
= &-2b_1^\dagger a_2\, a_3^\dagger b_4 - 2a_1^\dagger b_2\, b_3^\dagger a_4\\
&+2a_1^\dagger a_2\, b_3^\dagger b_4 + 2b_1^\dagger b_2\, a_3^\dagger a_4\\
= &\ 2(a_1^\dagger b_3^\dagger - b_1^\dagger a_3^\dagger)(a_2\, b_4 - b_2 a_4)
\end{aligned}$$

Now by inspection

$$P_{13}H_w = \frac{g}{4\sqrt{2}} \bar{\psi}_3 \gamma_\mu (1 + \gamma_5) \psi_2 \bar{\psi}_1 \gamma_\mu (1 + \gamma_5) \psi_4$$

$$= 2(a_3^\dagger b_1^\dagger - b_3^\dagger a_1^\dagger)(a_2 b_4 - b_2 a_4)$$

$$= -H_w$$

This is the desired result. It shows that the V - A interaction with maximum parity violation is a P_{13} invariant with eigenvalue -1. The minus one eigenvalue was obtained with the assumption that the different 4-spinors commute. If anticommutation relations are assumed to hold the eigenvalue is, of course, plus one.

In closing it should be noted that in the Cabibbo theory the approximate V - A structure of nuclear beta decay is built in empirically; and hence so is the P_{13} invariance of nuclear beta decay. In the following chapters an attempt will be made to construct a theory which incorporates P_{13} invariance as a basic assumption and at the same time overcomes the two objections raised against the Cabibbo theory; namely, (1) the Cabibbo theory does not determine the relative strengths of strangeness changing processes and strangeness conserving processes from "first principles," and (2) the Cabibbo theory when extended to non-leptonic processes leaves an asymmetry between baryons and leptons. No attempt will be made to retain the basic assumptions of the Cabibbo theory.

CHAPTER II

A REORDERING THEOREM FOR SEMISIMPLE COMPACT GROUPS

As was pointed out in the previous chapter the accepted V-A theory of beta-decay is a P_{13} invariant if the renormalization effects of the strong interactions are neglected. This P_{13} invariance of the bare V-A interaction is to a large extent ignored nowadays; however, one cannot help but wonder if it has some underlying significance. Considering this significance to lie in a self consistent analogy of particles Ahrens (52) has investigated the possible structures of the weak interaction as determined by permutation symmetry in isospace. A natural extension of that work in light of the success of the SU(3)-octet model for hadrons (53) is to investigate the permutation properties of biquadratic SU(3) scalars. This chapter is intended to lay the ground work for such investigations.

Since the Lorentz group and the special unitary group in three dimensions, SU(3), are structurally quite different, the existence of a reordering theorem of the Fierz type is questionable. It will be shown in this chapter that not only does such a theorem exist for SU(3) but it exists at least for all semisimple compact groups.

K. M. Case (51) has shown previously that a Fierz type reordering theorem exists for complex orthogonal groups. His proof requires the existence of a set of quantities $\Gamma(i)$ having commutation properties of the form

$$\Gamma(i)\,\Gamma(j) + \Gamma(j)\,\Gamma(i) = 2\,\delta_{ij}$$

Since such commutation rules do not occur for compact groups in general (SU(2) is an exception) Case's proof is not applicable to such groups. It is worth noting that Case's theorem and the reordering theorem presented here are essentially complementary and together they cover most groups of physical interest.

Before discussing the reordering theorem two examples of the P_{13} operation will be presented. These examples concern the reordering of the simplest nontrivial SU(2) and SU(3) biquadratic scalars. Next, as a preliminary to the statement and proof of the reordering theorem, the question of the reality of the Clebsch-Gordan (C-G) coefficients for semisimple groups will be considered. The general properties of the reordering matrix will be discussed in the last section of this chapter.

Examples

Two elementary examples of possible physical importance will be discussed in this section in order to become familiar with the P_{13} operation. Consider first the isoscalars formed from two nucleons and two antinucleons. There are two such scalars:

$$s = \bar{N}_1\, N_2\, \bar{N}_3\, N_4$$

and

$$v = \bar{N}_1\, \tau\, N_2 \cdot \bar{N}_3\, \tau\, N_4$$

where the N's are isodoublets, e.g. $N = \begin{pmatrix} p \\ n \end{pmatrix}$, and τ is the usual 2 x 2 isospin matrix vector. Expanding and assuming the various field operators to commute gives

$$P_{13}\ s \equiv \bar{N}_3\ N_2\ \bar{N}_1\ N_4 = \frac{1}{2}\ s + \frac{1}{2}\ v$$

and

$$P_{13} v = \frac{3}{2}\ s - \frac{1}{2}\ v$$

In this case and in the following SU(3) example the P_{24} and P_{13} operations are equivalent so that

$$P_{24}\ v = \frac{3}{2}\ s - \frac{1}{2}\ v\ , \text{ etc.}$$

In general P_{24} and P_{13} are equivalent up to a phase.

If the matrix α is defined by

$$P_{13} \begin{pmatrix} s \\ v \end{pmatrix} = \alpha \begin{pmatrix} s \\ v \end{pmatrix}$$

then clearly

$$\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Since the square of P_{13} leaves the scalars unchanged α^2 must be the unit matrix as is indeed the case. The P_{13} invariants are now easily seen to be $v + 3s$ and $v - s$ with eigenvalues $+1$ and -1 respectively (51).

Next consider the SU(3) scalars formed from two sakatons and two antisakatons (54). Again there are two scalars. This can be seen as follows. The sakatons belong to the irreducible representation (IR) 3 of SU(3) while the antisakatons belong to the contragredient IR 3^*. The IR's of SU(3) are labeled by their dimensions and an asterisk is used to denote the corresponding contragredient IR. The Kronecker product $3^* \times 3$ decomposes into the direct sum $1 + 8$. Hence

$$(3^* \times 3) \times (3^* \times 3) = (1 + 8) \times (1 + 8)$$

$$= (1 \times 1) + (1 \times 8) + (8 \times 1) + (8 \times 8)$$

Now in the decomposition of a direct product of two IR's of SU(3) a scalar will appear once and only once if and only if the product involves an IR and its contragredient (55). Furthermore for SU(3) an IR is equivalent to its contragredient if and only if its dimension is the cube of an integer. Thus it is clear that in the above expression one scalar appears in 1×1 and another in 8×8. These scalars are respectively

$$S = \bar{b}_1\, b_2\, \bar{b}_3\, b_4$$

and

$$V = \sum_i \bar{b}_1\, \lambda_i\, b_2\, \bar{b}_3\, \lambda_i\, b_4$$

where $b = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}$ is the sakaton and the λ's are Gell-Mann's matrices (23). Here the "bars" simply denote the complex-conjugate transpose.

Again expanding and assuming the various factors to commute gives

$$P_{13}\, S \equiv \bar{b}_3\, b_2\, \bar{b}_1\, b_4 = \frac{1}{3}\, S + \frac{1}{2}\, V$$

and

$$P_{13}\, V \equiv \sum_i \bar{b}_3 \lambda_i\, b_2\, \bar{b}_1\, \lambda_i\, b_4 = \frac{16}{9} S - \frac{1}{3} V$$

This can be written in matrix form as

$$P_{13} \begin{pmatrix} S \\ V \end{pmatrix} = \beta \begin{pmatrix} S \\ V \end{pmatrix}$$

where the reordering matrix β is given by

$$\beta = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{16}{9} & -\frac{1}{3} \end{pmatrix}$$

In this case the P_{13} invariants are $S + \frac{3}{8} V$ and $S - \frac{3}{4} V$ with eigenvalues $+1$ and -1 respectively.

Reality of the Clebsch-Gordan Coefficients

After considerable effort the commutation relations for the generators of a semisimple Lie group can be cast into the following canonical form (56, 57).

$$[H_i, H_j] = 0 \tag{1}$$

$$[H_i, E_\alpha] = \alpha_i\, E_\alpha \tag{2}$$

$$[E_\alpha, E_{-\alpha}] = \sum_i \alpha_i\, H_i \tag{3}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta}, \quad \text{if} \quad \alpha + \beta \neq 0 \tag{4}$$

The number of mutually commuting generators, H_i, is called the rank, ℓ, of the Lie algebra of the group while the total number of independent generators is called the order, r.

Equation 2 is conveniently thought of as an "eigenvector problem" where the E_α are simultaneous eigenvectors of the ℓ commuting generators, H_i, with eigenvalues, α_i. The eigenvalues are said to form a root vector (or simply a root)

$$\alpha = (\alpha_1, \alpha_2, \dots \alpha_\ell)$$

These root vectors are completely determined by the structure of the group. The notation $E_{\alpha+\beta}$ in equation 4 thus means that if $\alpha + \beta$ is a root then $E_{\alpha+\beta}$ is an eigenvector associated with that root and $N_{\alpha\beta} \neq 0$. On the other hand if $\alpha + \beta$ is not a root $N_{\alpha\beta}$ is zero.

By suitably normalizing the generators (which has already been assumed in equations 1 - 4) one obtains

$$\sum_\alpha \alpha_i \alpha_j = \delta_{ij} \tag{5}$$

and

$$N_{\alpha\beta} = -N_{-\alpha,-\beta} = \pm \sqrt{(k+1)(j+1)} \quad |\beta| \tag{6}$$

The sum in equation 5 is over all root vectors while the numbers j and k in equation 6 are the smallest positive integers such that

$$[E_{\alpha + j\beta}, E_\beta] = 0$$

and

$$[E_{\alpha - k\beta}, E_{-\beta}] = 0$$

respectively. The numbers j and k are completely determined by the roots and can be easily evaluated by inspection of the root diagram (56).

The N's also satisfy (56)

$$N_{\alpha\,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} \tag{7}$$

and the trivial relation

$$N_{\alpha\,\beta} = -N_{\beta\,\alpha} \tag{8}$$

In practice one uses equation 6 to select a suitable set of N's and then proceeds to find the matrices of the generators in a given representation with the aid of equations 3 and 4. When following this procedure the sign in equation 6 must be chosen so as to be consistent with equations 7 and 8 but is otherwise arbitrary.

One apparent motive for casting the commutation relations into the canonical form, although it has not been explicitly indicated in the literature, is that it allows one to choose the (matrix representatives of the) E_α real. Since by equation 2 the E_α are simply raising and lowering operators, the reality of the E_α determines the reality of the C-G coefficients associated with the group.

In order to show that the E_α can be chosen real consider a theorem due to Cartan and discussed by Racah (57) which proves that the eigenvectors of equation 2 are nondegenerate if the root vectors are nonzero. Thus by taking the transpose (T) and also the hermitian adjoint ($\dagger$) of the canonical commutation relations and making use of equation 6 one sees that $E_\alpha^{\,T}$ and $E_\alpha^{\,\dagger}$ are both proportional to $E_{-\alpha}$. Thus either

E_α^T or $E_\alpha^\dagger$ can be chosen equal to $E_{-\alpha}$. Choosing

$$E_\alpha^\dagger = E_{-\alpha} \tag{9}$$

and

$$E_\alpha^T = \lambda E_{-\alpha} \tag{10}$$

it only remains to be shown that the remaining constant of proportionality, λ, can be set equal to one for all α so that

$$E_\alpha^\dagger = E_\alpha^T = E_{-\alpha} \tag{11}$$

The equality in equation 9 is the conventional choice. However, no explicit statement concerning the equality in equation 10 nor of the freedom to establish equation 11 seems to appear in the literature. The H_i are only of secondary concern here since they can be taken to be hermitian and diagonal and hence real in all representations of the group. This in turn implies that the roots α are real (56). However, it is important to keep in mind that the H_i matrices along with the root vectors determine the representation of the group involved.

Equations 9 and 10 establish that E_α differs from a real matrix only by a phase which is the same for all the matrix elements of E_α. If one sets

$$E_\alpha = e^{i\varphi\alpha} E_\alpha'$$

where E_α' is real then E_α' automatically satisfies equations 2 and 3;

and equation 4 becomes

$$[E'_\alpha,\ E'_\beta] = N_{\alpha\beta}\, e^{i(\varphi_{\alpha+\beta} - \varphi_\alpha - \varphi_\beta)}\, E'_{\alpha+\beta}$$

The exponential is real and thus equal to ± 1 since the other factors are real. This is essentially the arbitrary sign in equation 6. Hence the exponential can be absorbed into $N_{\alpha\beta}$. In fact if one defines

$$N'_{\alpha\beta} = N_{\alpha\beta}\, e^{i(\varphi_{\alpha+\beta} - \varphi_\alpha - \varphi_\beta)}$$

it is easily seen that the $N'_{\alpha\beta}$ satisfy equations 6 - 8 if the $N_{\alpha\beta}$ do. This then establishes that the generators can be chosen real without disrupting the canonical commutation relations in any representation which diagonalizes the H_i.

Now consider the C-G coefficients $\langle j_1\,\mu_1\,\nu_1;\ j_2\,\mu_2\,\nu_2 | j(\gamma)\,\mu\nu\rangle$ defined by

$$|(j_1\ j_2) j(\gamma)\ \mu\nu\rangle = \sum_{\substack{\mu_1\nu_1 \\ \mu_2\nu_2}} \langle j_1\mu_1\nu_1;\ j_2\mu_2\nu_2 | j(\gamma)\mu\nu\rangle\ |j_1\mu_1\nu_1\rangle\, |j_2\mu_2\nu_2\rangle \qquad (12)$$

where (1) j_1, j_2 and j label IR's of the group,

(2) the γ's are multiplicity labels which must be used if an IR can appear more than once in the decomposition of the Kronecker product of two IR's of the group (i.e. the group is not simply reducible (57, 58)),

(3) ν_1, ν_2 and ν are vectors whose components are the eigenvalues

of H_i which partially label the states of the IR, i.e. the ν's are weight vectors (56, 57), and

(4) the μ's are other labels which are required to distinguish between states with the same weight (57, 59).

Here as in the remainder of this thesis $|j\mu\nu\rangle$ denotes the state labeled by μ and ν in the IR labeled by j. If the state has been obtained by coupling two other sets of states as in equation 12 these states will be placed in parentheses before the j while the multiplicity label, if there is one, will be placed in parentheses after the j.

The states $|(j_1, j_2)j(\gamma)\mu\,\nu\rangle$ form a complete orthonormal set as do the $|j\mu\nu\rangle$. Thus

$$\langle(j_1\ j_2)j(\gamma)\mu\nu|(j_1\ j_2)k(\beta)\omega\,\rho\rangle = \delta_{jk}\delta_{\gamma\beta}\delta_{\mu\omega}\delta_{\nu\rho} \tag{13a}$$

and

$$\langle j\,\mu\,\nu|k\omega\,\rho\rangle = \delta_{jk}\,\delta_{\mu\omega}\,\delta_{\nu\rho} \tag{13b}$$

If $|j_1\,\mu_1\,\nu_1\rangle$ and $|j_2\,\mu_2\,\nu_2\rangle$ are assumed to be in different spaces the raising and lowering operators for $|(j_1\ j_2)j(\gamma)\ \mu\ \nu\rangle$ can be written

$$E_\alpha = E_\alpha(1) + E_\alpha(2) \tag{14}$$

where $E_\alpha(i)$ operates only on $|j_i\ \mu_i\ \nu_i\rangle$. Now from equation (12)

$$E_\alpha(i)|j_i\,\mu_i\,\nu_i\rangle = \sum_{\mu_i'} \lambda_\alpha(\mu_i', j_i\mu_i\nu_i)|j_i\mu_i'\nu_i + \alpha\rangle\ ;\ i = 1,2 \tag{15}$$

where the λ_α are real since they are just the matrix elements of $E_\alpha(i)$. Operating on equation 12 with E_α thus gives

$$\sum_{\mu'} \lambda_\alpha(\mu', j\mu\nu)|(j_1j_2)j(\gamma)\mu'\nu+\alpha\rangle = \sum_{\substack{\mu_1\nu_1\\ \mu_2\nu_2}} \langle j_1\mu_1\nu_1;j_2\mu_2\nu_2|j(\gamma)\mu\nu\rangle \tag{16}$$

$$\times\left[\sum_{\mu_1'} \lambda_\alpha(\mu_1', j_1\mu_1\nu_1)|j_1\mu_1'\nu_1+\alpha\rangle|j_2\mu_2\nu_2\rangle + \sum_{\mu_2'} \lambda_\alpha(\mu_2', j_2\mu_2\nu_2)|j_1\mu_1\nu_1\rangle|j_2\mu_2'\nu_2+\alpha\rangle\right]$$

In the first term on the right hand side replace the dummy index ν_1 by $\nu_1 - \alpha$ and interchange the dummy indices μ_1 and μ_1'. Similarly in the second term replace ν_2 by $\nu_2 - \alpha$ and interchange μ_2 and μ_2'. Equation 16 then becomes

$$\sum_{\mu'} \lambda_\alpha(\mu', j\mu\nu)|(j_1j_2)j(\gamma)\mu'\nu+\alpha\rangle \tag{17}$$

$$= \sum_{\substack{\mu_1\nu_1\\ \mu_2\nu_2}} \left[\sum_{\mu_1'} \lambda_\alpha(\mu_1, j_1\mu_1'\nu_1-\alpha)\langle j_1\mu_1'\nu_1-\alpha;j_2\mu_2\nu_2|j(\gamma)\mu\nu\rangle\right.$$

$$\left. + \sum_{\mu_2'} \lambda_\alpha(\mu_2, j_2\mu_2'\nu_2-\alpha)\langle j_1\mu_1\nu_1;j_2\mu_2'\nu_2-\alpha|j(\gamma)\mu\nu\rangle\right]$$

$$\times |j_1\mu_1\nu_1\rangle|j_2\mu_2\nu_2\rangle$$

Solution of equation 17 for the $|(j_1j_2)j(\gamma)\mu'\nu+\alpha\rangle$ determines a recursion relation for the C-G coefficients $\langle j_1\mu_1\nu_1;j_2\mu_2\nu_2|j(\gamma)\mu'\nu+\alpha\rangle$ in terms of the λ_α which are real and the coefficients

$$\langle j_1\mu_1\nu_1;j_2\mu_2\nu_2|j(\gamma)\mu\nu\rangle$$

Clearly the C-G coefficients for the weight $\nu + \alpha$ will be real if those for ν are. This will be the case if the C-G coefficient for the state in j with highest weight (56) (i.e. the state in the IR whose weight has the largest positive first component) is chosen real. The phase of this "highest state" can always be adjusted so that this is the case.

The Reordering Theorem

Let ψ and φ be column matrices whose elements are the orthonormal states of the IR j of a semisimple compact group G. Similarly let the elements of Ψ and Φ be the orthonormal basis states of the IR j^* of G. Thus, for example, the $(\mu\nu)$ component of ψ is given by $\psi(\mu\nu) = |j\mu\nu\rangle$. Now assume that the IR K appears in the decomposition of the Kronecker product $j^* \times j$. The IR K^* must also appear in the decomposition since $j \times j^*$ is equivalent to $j^* \times j$. Thus one may, for example, construct from Ψ and ψ quantities which transform according to the IR K^* and from Φ and φ quantities which transform according to the IR K. These can be expressed with the aid of the C-G coefficients as

$$\sum_{\substack{\mu_1\nu_1\\ \mu_2\nu_2}} \langle j^*\mu_1\nu_1;j\,\mu_2\nu_2|K^*(\epsilon)\mu\nu\rangle\, \Psi(\mu_1\nu_1)\,\psi(\mu_2\nu_2)$$

and

$$\sum_{\substack{\mu_1\nu_1\\ \mu_2\nu_2}} \langle j^*\,\mu_1\nu_1;j\,\mu_2\nu_2|K(\delta)\mu\nu\rangle\, \Phi(\mu_1\nu_1)\,\varphi(\mu_2\nu_2)$$

respectively. From these quantities it is possible to construct a quantity denoted by $S^{j}_{K(\varepsilon\delta)}(\Psi\psi, \Phi\varphi)$ which transforms as a scalar under G.

The P_{24} operation (which is equivalent to P_{13} up to a phase) in this case interchanges ψ and φ. Thus by definition

$$P_{24}\, S^{j}_{K(\varepsilon\delta)}(\Psi\psi, \Phi\varphi) = S^{j}_{K(\varepsilon\delta)}(\Psi\varphi, \Phi\psi) \qquad (18)$$

The reordering theorem is then expressed by the equation

$$P_{24} S^{j}_{K(\varepsilon\delta)}(\Psi\psi, \Phi\varphi) = \sum_{J\beta\gamma} \alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)}\, S^{j}_{J(\beta\gamma)}(\Psi\psi, \Phi\varphi) \quad (19)$$

where the sum is a double sum over all IR's occurring in the decomposition of $j^{*} \times j$. Equation 19 is conveniently written in matrix form as

$$P_{24}\, S^{j} = \alpha^{j}\, S^{j} \qquad (20)$$

where S^{j} is a column matrix whose elements are the scalars $S^{j}_{J(\beta\alpha)}$ and α^{j} is the "reordering matrix" whose elements are the numbers $\alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)}$.

The proof of the reordering theorem depends on only four conditions:

(1) the "completeness" of the C-G coefficients

(2) the reality of the C-G coefficients

(3) the symmetry of the C-G coefficients under interchange of the two factor states $|j_1\mu_1\nu_1\rangle$ and $|j_2\mu_2\nu_2\rangle$ in equation 12, and

(4) the uniqueness of the construction of a scalar from an IR and its contragredient.

Condition 2 is satisfied by semisimple groups as was shown in the previous section, condition 4 is satisfied by compact groups (see ref. 55, pp. 147 and 317) and conditions 1 and 3 are satisfied in general (55) (at least whenever it is meaningful to define C-G coefficients).

Since the C-G coefficients are real and transform an orthonormal basis into an orthonormal basis, they form a real orthogonal matrix. Definition 12 and its inverse then yield what will be referred to in this thesis as the C-G orthogonality and completeness relations.

Orthogonality:

$$\sum_{m_1 m_2} \langle j_1 m_1; j_2 m_2 | J(\gamma) M \rangle \langle j_1 m_1; j_2 m_2 | K(\beta) N \rangle = \delta_{JK} \delta_{\beta\gamma} \delta_{MN} \tag{21}$$

Completeness:

$$\sum_{J\gamma M} \langle j_1 m_1; j_2 m_2 | J(\gamma) M \rangle \langle j_1 m_3; j_2 m_4 | J(\gamma) M \rangle = \delta_{m_1 m_3} \delta_{m_2 m_4} \tag{22}$$

where the m's etc. represent all of the appropriate state labels within an IR.

The symmetry property mentioned in condition 3 arises from the equivalence of $j_1 \times j_2$ and $j_2 \times j_1$ for arbitrary IR's j_1 and j_2. This requires that

$$\langle j_2 m_2; j_1 m_1 | J(\gamma) M \rangle = \xi_1 (j_2 j_1 J(\gamma)) \langle j_1 m_1; j_2 m_2 | J(\gamma) M \rangle \tag{23}$$

where ξ_1 is a real phase factor which is independent of the state labels m_1, m_2 and M.

Since only those C-G coefficients will occur in the following which couple an IR and its contragredient it is convenient to introduce the matrices $\langle jJ(\Upsilon)M \rangle$ whose elements are defined by

$$\langle jJ(\Upsilon)M \rangle_{ab} = \langle j^*a;\ jb \,|J(\Upsilon)M \rangle \tag{24}$$

where a and b are all of the appropriate state labels. The orthogonality, completeness and symmetry relations then become respectively

$$\mathrm{Tr}(\langle jJ(\Upsilon)M \rangle \langle jK(\beta)N \rangle^T) = \delta_{JK}\delta_{\beta\Upsilon}\delta_{MN} \tag{25}$$

$$\sum_{J\Upsilon M} \langle jJ(\Upsilon)M \rangle_{m_1 m_2} \langle jJ(\Upsilon)M \rangle_{m_3 m_4} = \delta_{m_1 m_3}\delta_{m_2 m_4} \tag{26}$$

and

$$\langle j^*J(\Upsilon)M \rangle_{m_2 m_1} = \xi_1(jj^*J(\Upsilon)) \langle jJ(\Upsilon)M \rangle_{m_1 m_2} \tag{27}$$

Substituting equation 27 into equation 26 gives

$$\sum_{J\Upsilon M} \frac{1}{\xi_1(jj^*J(\Upsilon))} \langle j^*J(\Upsilon)M \rangle_{m_2 m_1} \langle jJ(\Upsilon)M \rangle_{m_3 m_4} = \delta_{m_1 m_3}\delta_{m_2 m_4} \tag{28}$$

Now let θ and θ' be two arbitrary $n_j \times n_j$ matrices where n_j is the dimension of the IR j and multiply equation 28 by $(\psi^T\theta)_{m_2}\varphi_{m_4}\Phi^T_{m_3}(\theta'\psi)_{m_1}$. Summing on the m's gives

$$\Psi^T \theta \varphi \Phi^T \theta' \psi = \sum_{J\Upsilon M} \frac{1}{\xi_1(jj^*J(\Upsilon))} \Psi^T\theta \langle j^*J(\Upsilon)M \rangle \theta'\psi \,\Phi^T \langle jJ(\Upsilon)M \rangle \varphi \tag{29}$$

Equation 29 is the analog of equation 81 in Case's paper (51). Choosing $\theta = \langle j\ K^*(\varepsilon)\ -N \rangle$ and $\theta' = \langle j\ K(o)N \rangle$, multiplying by the "1 - j symbol"

$\langle K10 \rangle_{-NN}$ and summing on N gives

$$\sum_{N} \Psi^T \langle jK^*(\varepsilon) - N \rangle \varphi \langle K10 \rangle_{-NN} \Phi^T \langle jK(\delta)N \rangle \psi \tag{30}$$

$$= \sum_{J\gamma M} \sum_{N} \frac{\langle K10 \rangle_{-NN}}{\xi_1(jj^*J(\gamma))} \Psi^T \langle jK^*(\varepsilon) - N \rangle \langle j^*J(\gamma)M \rangle \langle jK(\delta)N \rangle \psi \, \Phi^T \langle jJ(\gamma)M \rangle \varphi$$

The $1-j$ symbol $\langle K10 \rangle_{-NN}$ is simply the appropriate set of C-G coefficients for combining states which transform according to the IR's K^* and K to give a state which transforms as a scalar (which is labeled by 1 and whose single component is labeled by 0). The symbol $-N$ labels that state which must be combined with the state labeled by N in forming a scalar quantity

Now the left side of equation 30 is a scalar under the group. Hence the right side is also a scalar. Since the quantity $\Phi^T \langle jJ(\gamma)M \rangle \varphi$ transforms according to the IR J, the scalar character of equation 30 requires the remainder of the expression under the sum on M to be the appropriate $1-j$ symbol times a quantity which transforms as the $-M$ component of the IR J^*. This is a direct consequence of condition 4. Hence

$$\frac{1}{\xi_1(jj^*J(\gamma))} \sum_{N} \langle K10 \rangle_{-NN} \langle jK^*(\varepsilon) - N \rangle \langle j^*J(\gamma)M \rangle \langle jK(\delta)N \rangle \tag{31}$$

$$= \sum_{\beta} \alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)} \langle J10 \rangle_{-MM} \langle jJ^*(\beta) - M \rangle$$

Equation 30 thus becomes

$$S^j_{K(\varepsilon\delta)}(\Psi\varphi, \Phi\psi) = \sum_{J\beta\gamma} \alpha^j_{K(\varepsilon\delta)J(\beta\gamma)} \, S^j_{J(\beta\gamma)}(\Psi\psi, \Phi\varphi) \tag{32}$$

where

$$S^j_{K(\varepsilon\delta)}(\Psi\varphi, \Phi\psi) = \sum_N \Psi^T \langle jK^*(\varepsilon) -N\rangle \varphi \langle K10\rangle_{-NN} \Phi^T \langle jK(\delta)N\rangle \psi \tag{33}$$

Equation 32 is simply the reordering theorem of equations 18 and 19.

Properties of the Reordering Matrix

Since the square of P_{24} leaves the scalars unaltered, equation 20 implies that

$$(\alpha^j)^2 = 1 \tag{34}$$

Thus α^j is its own inverse.

Now consider the following inner product of the scalars in equation 33

$$(S^j_{K(\varepsilon\delta)}(\Psi\varphi, \Phi\psi), S^j_{J(\beta\gamma)}(\Psi\varphi, \Phi\psi)) \tag{35}$$

$$= \sum_{MN} \sum_{\substack{abcd\\efgh}} \langle jK^*(\varepsilon) -N\rangle_{ab} \langle K10\rangle_{-NN} \langle jK(\delta)N\rangle_{cd} \langle jJ^*(\beta) -M\rangle_{ef} \langle J10\rangle_{-MM}$$

$$\times \; \langle jJ(\gamma)M\rangle_{gh} (\Psi(a), \Psi(e))(\varphi(b), \varphi(f))(\Phi(c), \Phi(g))(\psi(d), \psi(h))$$

This is the usual inner product for composite systems encountered in quantum mechanics. Now by virtue of equation 13 (recalling that the state labels have been compressed to a single index) the last four factors give

Kronecker deltas. For example

$$(\Psi(a),\ \Psi(e)) = \delta_{ae}$$

Thus equation 35 becomes

$$S^{j}_{K(\varepsilon\delta)}(\Psi\varphi, \Phi\psi)\ ,\ S^{j}_{J(\beta\gamma)}(\Psi\varphi, \Phi\psi) \tag{36}$$

$$= \sum_{MN} \sum_{abcd} \left\langle jK^{*}(\varepsilon) - N \right\rangle_{ab} \left\langle K1O \right\rangle_{-NN} \left\langle jK(\delta)N \right\rangle_{cd} \left\langle jJ^{*}(\beta) - M \right\rangle_{ab}$$

$$\times \left\langle J1O \right\rangle_{-MM} \left\langle jJ(\gamma)M \right\rangle_{cd}$$

$$= \delta_{JK}\ \delta_{\varepsilon\beta}\ \delta_{\gamma\delta}$$

where the last equality follows after several applications of equation 25.

Equation 36 demonstrates that the reordering matrix transforms one orthonormal set of quantities into another orthonormal set. Hence the α^{j} must be unitary. Furthermore the α^{j} are real by virtue of equation 31 and the fact that the C-G coefficients are real. Combining these results with equation 34 implies that the reordering matrices are real, symmetric and orthogonal. Thus

$$\alpha^{j} = (\alpha^{j})^{*} = (\alpha^{j})^{T} = (\alpha^{j})^{-1} \tag{37}$$

where * means complex conjugate.

An additional consequence of the orthogonality of the reordering matrix is the implication that there are the same number of P_{24} (or P_{13}) invariants as there are scalars in the decomposition of the Kronecker product $(j^{*} \times j) \times (j^{*} \times j)$.

CHAPTER III

APPLICATION TO SU(3)

Generalized Racah Coefficients

In this first section the definition and structure of the Racah W-coefficients for an arbitrary semisimple compact group G will be discussed. The definition of the Racah W-coefficients to be used here is analogous to that given by Rose (60) for the corresponding SU(2) quantities.

Consider the coupling according to equation II-12* of three orthonormal sets of quantities $|j_1\, m_1\rangle$, $|j_2\, m_2\rangle$ and $|j_3\, m_3\rangle$ which transform according to the IR's j_1 , j_2 and j_3 respectively of G. One could couple first $|j_1\, m_1\rangle$ and $|j_2\, m_2\rangle$ to give $|(j_1 j_2)j'(\gamma')m'\rangle$ and then couple this to $|j_3\, m_3\rangle$ to give $|\big((j_1\, j_2)j'(\gamma')j_3\big)j(\gamma)m\rangle$ or as one alternative couple $|j_2\, m_2\rangle$ and $|j_3\, m_3\rangle$ to give $|(j_2\, j_3)j''(\gamma'')m''\rangle$ and then couple this to $|j_1\, m_1\rangle$ to give $|\big(j_1(j_2\, j_3)j''(\gamma'')\big)j(\beta)m\rangle$. The two sets of orthonormal quantities $|\big((j_1\, j_2)j'(\gamma')j_3\big)j(\gamma)m\rangle$ and $|\big(j_1(j_2\, j_3)j''(\gamma'')\big)j(\beta)m\rangle$ span the same space and hence must be related by a unitary transformation.

$$|\big((j_1\, j_2)j'(\gamma')j_3\big)j(\gamma)m\rangle \tag{1}$$

$$= \sum_{j''\gamma''\beta}{}' \sqrt{n_{j''}n_{j'}}\; W\big(j_1 j_2(\gamma\beta)j_3;j'(\gamma')j''(\gamma'')\big) \times |\big(j_1(j_2 j_3)j''(\gamma'')\big)j(\beta)m\rangle$$

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*Important equations are numbered consecutively within each chapter. Thus, for example, equation 12 refers to equation 12 of the current chapter while equation II-12 refers to equation 12 of Chapter II.

This relation defines the W-coefficients for G. It is important to note that since the quantities in equation 1 belong to the same m the W-coefficients are independent of m. (In this connection see ref. 26, pp. 115 and 298.)

Using equation II-12 to express the composite quantities in terms of $|j_1\, m_1\rangle$, $|j_2\, m_2\rangle$ and $|j_3\, m_3\rangle$ gives

$$\sum_{\substack{m_1 m_2 \\ m_3 m'}} \langle j'm';j_3m_3|j(\gamma)m\rangle \langle j_1m_1;j_2m_2|j'(\gamma')m'\rangle\ |j_1m_1\rangle\, |j_2m_2\rangle\, |j_3m_3\rangle$$

$$= \sum_{j''\gamma''\beta} \sum_{\substack{m_1 m_2 \\ m_3 m''}} \sqrt{n_{j''}n_{j'}}\ W(j_1j_2j(\gamma\beta)j_3;j'(\gamma')j''(\gamma''))$$

$$\times \langle j_1m_1;j''m''|j(\beta)m\rangle \langle j_2m_2;j_3m_3|j''(\gamma'')m''\rangle\ |j_1m_1\rangle\, |j_2m_2\rangle$$

$$\times\ |j_3m_3\rangle \tag{2}$$

The orthonormality of the products $|j_1\, m_1\rangle\, |j_2\, m_2\rangle\, |j_3\, m_3\rangle$ requires their coefficients on the two sides of equation 2 to be equal. Thus

$$\sum_{m'} \langle j'm';j_3m_3|j(\gamma)m\rangle \langle j_1m_1;j_2m_2|j'(\gamma')m'\rangle \tag{3}$$

$$= \sum_{\substack{j''\gamma'' \\ m''\beta}} \sqrt{n_{j''}n_{j'}}\ W(j_1j_2j(\gamma\beta);j'(\gamma')j''(\gamma''))\ \langle j_1m_1;j''m''|j(\beta)m\rangle$$

$$\times\ \langle j_2m_2;j_3m_3|j''(\gamma'')m''\rangle$$

Using the orthogonality of the C-G coefficients twice then gives

$$W\left(j_1 j_2 j(\gamma\beta) j_3 ; j'(\gamma') j''(\gamma'')\right) \tag{4}$$

$$= (n_{j''} n_{j'})^{-\frac{1}{2}} \sum_{\substack{m_1 m_2 m_3 \\ m' m''}} \left\langle j'm' ; j_3 m_3 | j(\gamma) m \right\rangle \left\langle j_1 m_1 ; j_2 m_2 | j'(\gamma') m' \right\rangle$$

$$\times \left\langle j_2 m_2 ; j_3 m_3 | j''(\gamma'') m'' \right\rangle \left\langle j_1 m_1 ; j'' m'' | j(\beta) m \right\rangle$$

for all values of m in the IR j. This result is, of course, the same as in the SU(2) case with two generalizations: (1) the multiplicity labels have been introduced, and (2) the state labels and corresponding coupling rules may now be more complicated than in the SU(2) case.

Specializing G to the group SU(3) one can split the SU(3) C-G coefficients in the customary way into an SU(2) C-G coefficient and an "isoscalar factor" (24):

$$\left\langle j_1 m_1 ; j_2 m_2 | j(\gamma) \right\rangle = \left(j_1 I_1 Y_1 ; j_2 I_2 Y_2 | j(\gamma) I Y \right) \left\langle\!\left\langle I_1 I_{1z} ; I_2 I_{2z} | I I_z \right\rangle\!\right\rangle \tag{5}$$

where the double "bra-kets," $\langle\!\langle \ldots | \ldots \rangle\!\rangle$ denote the SU(2) C-G coefficients and the quantities $(\ldots|\ldots)$ are the isoscalar factors. The state labels have been chosen to be the "isospin," "z-component of isospin" and "hypercharge." That is, $m \equiv (I, I_z, Y)$, etc. Making the splitting for SU(3) in equation 4 and recombining the SU(2) C-G coefficients according to the same equation gives

$$W(j_1 j_2 j(\gamma\beta) j_3; j'(\gamma') j''(\gamma'')) = (n_{j''} n_{j'})^{-\frac{1}{2}} \sum_{\substack{I_1 I_2 I_3 \\ I' I''}} \sum_{\substack{Y_1 Y_2 Y_3 \\ Y' Y''}} \tag{6}$$

$$\sqrt{(2I''+1)(2I'+1)}\; W_2(I_1 I_2 I I_3; I' I'') (j' I' Y'; j_3 I_3 Y_3 | j(\gamma) I Y)$$

$$\times\ (j_1 I_1 Y_1; j_2 I_2 Y_2 | j'(\gamma') I' Y')\,(j_2 I_2 Y_2; j_3 I_3 Y_3 | j''(\gamma'') I'' Y'')$$

$$\times\ (j_1 I_1 Y_1; j'' I'' Y'' | j(\beta) I Y)$$

where W_2 denotes the SU(2) Racah coefficients.

The additivity of hypercharge can be used to evaluate the sums on Y_1, Y_2 and Y_3. Furthermore there is a unique state in every IR of SU(3) for which $I = 0$ (see ref. 61, pp. 26-28). This state can be used to simplify equation 6 since the SU(3) W-coefficients are independent of the choice of I and Y (provided, of course, that a state with the labels I and Y exists in the IR j). This gives

$$W(j_1 j_2 j(\gamma\beta) j_3; j'(\gamma') j''(\gamma'')) = (n_{j''} n_{j'})^{-\frac{1}{2}} \sum_{I_1 I_2 I_3} \sum_{Y' Y''} \tag{7}$$

$$\sqrt{(2I_1+1)(2I_3+1)}\; W_2(I_1 I_2 0 I_3; I_3 I_1) (j' I_3 Y'; j_3 I_3 Y-Y' | j(\gamma) 0 Y)$$

$$\times\ (j_1 I_1 Y-Y''; j_2 I_2 Y'+Y''-Y | j'(\gamma') I_3 Y')$$

$$\times\ (j_2 I_2 Y'+Y''-Y; j_3 I_3 Y-Y' | j''(\gamma'') I_1 Y'')\,(j_1 I_1 Y-Y''; j'' I_1 Y'' | j(\beta) 0 Y)$$

In obtaining this expression the sums on I' and I'' were evaluated by noting that in order to obtain $I = 0$ it is necessary that $I' = I_3$ and $I'' = I_1$.

The remaining W_2 coefficient is well known in the literature

(26,62). It is given by

$$W_2(I_1I_2OI_3;I_3I_1) = \frac{\delta(I_1,I_3,I_2)}{\sqrt{(2I_1+1)(2I_3+1)}}$$

where

$$\delta(I_1,I_3,I_2) = 1 \qquad \text{if } I_1 + I_2 + I_3 \text{ is an integer}$$

$$\text{and } I_1 + I_3 \geq I_2 \geq |I_1 - I_3|$$

$$= 0 \qquad \text{otherwise.}$$

Thus

$$W\left(j_1j_2j(\gamma\beta)j_3;j'(\gamma')j''(\gamma'')\right) = (n_{j''}n_{j'})^{-\frac{1}{2}} \sum_{I_1I_2I_3} \sum_{Y'Y''}{}' \tag{9}$$

$$\times\ \delta(I_1,I_3,I_2)\left(j'I_3Y';j_3I_3Y-Y'|j(\gamma)OY\right)$$

$$\times\ \left(j_1I_1Y-Y'';j_2I_2Y'+Y''-Y|j'(\gamma')I_3Y'\right)\ \left(j_2I_2Y'+Y''-Y;j_3I_3Y-Y'|j''(\gamma'')I_1Y''\right)$$

$$\times\ \left(j_1I_1Y-Y'';j''I_1Y''|j(\beta)OY\right)$$

This expression can now be used to evaluate the simpler SU(3) Racah coefficients using tables of isoscalar factors (see, e.g., ref. 24).

Specialization to the Eightfold Way

A relationship between the SU(3) reordering matrices associated with the octet model of hadrons and certain SU(3) W-coefficients will be established in this section. Such a relation would follow immediately in the SU(2) case where the extensive Racah algebra of 6-j and 9-j symbols is available (60,62).

By multiplying equation II-31 by $[\xi_1(jj^*J^*(\beta))\langle J10\rangle_{-MM}]^{-1}$ x $\langle j^*J^*(\beta)-M\rangle$ and taking the trace the following expression for the matrix elements of α^j is obtained.

$$\alpha^j_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{1}{\xi_1(jj^*J^*(\beta))\xi_1(jj^*J(\gamma))} \sum_N \frac{\langle K10\rangle_{-NN}}{\langle J10\rangle_{-MM}} \tag{10}$$

$$\times \mathrm{Tr}[\langle jK^*(\varepsilon)-N\rangle \langle j^*J(\gamma)M\rangle\langle jK(\delta)N\rangle\langle j^*J^*(\beta)-M\rangle]$$

Now in the octet model the hadrons are assigned to the various IR's of SU(3) which occur in the decompositions of the Kronecker products of the 8-dimensional IR of SU(3) with itself. For all of these IR's the hypercharge is integral. Because of this one can introduce a particularly convenient phase convention due to de Swart (24) relating the states of the IR's j and j^*:

$$|j^*m\rangle = \langle j-m|(-1)^{\bar{m}} \tag{11}$$

where $\langle j-m| = |j-m\rangle^*$ (* denotes complex conjugate) and $\bar{m} = I_z+Y/2$.

Using this phase convention de Swart derived the following symmetry properties of the SU(3) C-G coefficients.

$$\langle j_1m_1;j_2m_2|j_3(\gamma)m_3\rangle = \xi_2(j_1j_2j_3(\gamma))(-1)^{\bar{m}_1}\left(\frac{n_3}{n_2}\right)^{\frac{1}{2}} \tag{12}$$

$$\times \langle j_1m_1;j_3^*-m_3|j_2^*(\gamma')-m_2\rangle$$

$$\langle j_1 m_1 ; j_2 m_2 | j_3(\gamma) m_3 \rangle = \xi_3 \left(j_1 j_2 j_3(\gamma) \right) \langle j_1^* -m_1 ; j_2^* -m_2 | j_3^*(\gamma) -m_3 \rangle \tag{13}$$

The ξ's are real phase factors which are independent of the state labels m_1 , m_2 and m. The validity of equation 12 is questionable whenever there is more than one value of γ . However de Swart has established explicitly that it is satisfied for all cases of interest in this thesis.

These symmetry relations can now be used to rearrange equation 10. Several phase factors will be introduced by the symmetry relations during this rearrangement process. These phase factors will be denoted collectively by χ . Of course, the value of X will change from step to step but only the phase factor for the final expression will be recorded.

Indicating explicitly the sums on the matrix indices, changing the dummy index N to -N and applying equation 13 to the last two factors in the sum equation 10 becomes

$$a^{j}_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{\chi}{\langle J^* -M ; JM | 10 \rangle} \sum_{N} \sum_{abcd} \langle K^* N ; K -N | 10 \rangle \langle j^* a ; jb | K^*(\varepsilon) N \rangle$$

$$\times \langle jb ; j^* c | J(\gamma) M \rangle \langle j -c ; j^* -d | K^*(\delta) N \rangle \langle j^* -d ; j -a | J(\beta) M \rangle \tag{14}$$

Applying equation 12 to the second and fourth factors in the sum gives

$$a^{j}_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{\chi}{\langle J^* -M ; JM | 10 \rangle} \left(\frac{n_K}{n_j} \right) \sum_{N} \sum_{abcd} (-1)^{\bar{a}-\bar{c}} \langle K^* N ; K -N | 10 \rangle \tag{15}$$

$$\langle j^* a ; K -N | j^*(\lambda) -b \rangle \langle jb ; j^* c | J(\gamma) M \rangle \langle j -c ; K -N | j(\lambda') d \rangle \langle j^* -d ; j -a | J(\beta) M \rangle$$

where λ and λ' are such that

$$\langle j^*a;jb|K^*(\varepsilon)N\rangle = \xi_2|j^*jK^*(\varepsilon)|\ (-1)^{\bar{a}} \left(\frac{n_K}{n_j}\right)^{\frac{1}{2}} \langle j^*a;K-N|j^*(\lambda)-b\rangle$$

$$\langle j-c;j^*-d|K^*(\delta)N\rangle = \xi_2|\ jj^*K^*(\delta)|\ (-1)^{-\bar{c}} \left(\frac{n_K}{n_j}\right)^{\frac{1}{2}} \langle j-c;K-N|j(\lambda')d\rangle$$

Now using equation II-23 on the last two factors gives

$$\alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{X}{\langle J^*-M;JM|10\rangle} \left(\frac{n_K}{n_j}\right) \sum_{N} \sum_{abcd} (-1)^{\bar{a}-\bar{c}} \tag{16}$$

$$\times \ \langle K^*N;K-N|10\rangle \langle j^*a;K-N|j^*(\lambda)-b\rangle$$

$$\times \ \langle jb;j^*c|J(\gamma)M\rangle \ \langle K-N;j-c|j(\lambda')d\rangle \ \langle j-a;j^*-d|J(\beta)M\rangle$$

Finally using equation 13 on the second and fourth factors in the sum of equation 16 gives

$$\alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{X}{\langle J^*-M;JM|10\rangle} \left(\frac{n_K}{n_j}\right) \sum_{N} \sum_{abcd} (-1)^{\bar{a}-\bar{c}} \tag{17}$$

$$\times \ \langle K^*N;K-N|10\rangle \ \langle j-a;K^*N|j(\lambda)b\rangle$$

$$\times \ \langle jb;j^*c|J(\gamma)M\rangle \ \langle K^*N;j^*c|j^*(\lambda')-d\rangle \ \langle j-a;j^*-d|J(\beta)M\rangle$$

Now consider the factor $(-1)^{\bar{a}-\bar{c}} \dfrac{\langle K^*N;K-N|10\rangle}{\langle J^*-M;JM|10\rangle}$. According to deSwart (24)

$$\langle J^*-M;JM|10\rangle = \frac{(-1)^{\overline{M^*_H}-M}}{\sqrt{n_j}}$$

where M_H^* is the highest weight in J^*. Thus

$$(-1)^{\bar{a}-\bar{c}} \frac{\langle K^*N;K-N|10\rangle}{\langle J^*-M;JM|10\rangle} = (-1)^{\bar{a}-\bar{c}+\bar{N}+\bar{M}}(-1)^{\overline{N_H^*}-\overline{M_H^*}} \left(\frac{n_J}{n_K}\right)^{\frac{1}{2}} \tag{18}$$

(N_H^* is the highest weight in the IR K^*.) Clearly the summand of equation 17 gives a non-zero contribution only when $\bar{b} + \bar{c} = \bar{M}$ and $-\bar{a} + \bar{N} = \bar{b}$ or equivalently when $\bar{N} - \bar{a} + \bar{c} = \bar{M}$. Thus recalling that the barred quantities are integers in the octet model it follows that

$(-1)^{\bar{a}-\bar{c}+\bar{N}+\bar{M}} = (-1)^{(\bar{N}-\bar{a}+\bar{c})+\bar{M}} = (-1)^{2\bar{M}} = 1$. Hence

$$\alpha^j_{K(\varepsilon\delta)J(\beta\gamma)} = \frac{\chi\sqrt{n_J n_K}}{n_j} \sum_N \sum_{abcd} \langle j-a;K^*N|j(\lambda)b\rangle \tag{19}$$

$$\times \langle jb;j^*c|J(\gamma)M\rangle \langle K^*N;j^*c|j^*(\lambda')-d\rangle \langle j-a;j^*-d|J(\beta)M\rangle$$

In equation 19 the value of χ is given by

$$\chi = (-1)^{\overline{M_H^*}+\overline{N_H^*}} \xi_1\{jKj(\lambda')\}\ \xi_1\{j^*jJ(\beta)\}\ \xi_1\{jj^*J^*(\beta)\}\ \xi_1\{jj^*J(\gamma)\}$$

$$\times\ \xi_2\{j^*jK^*(\varepsilon)\}\ \xi_2\{jj^*K^*(\delta)\}\ \xi_3\{j^*jK(\delta)\}\xi_3\{jj^*J^*(\beta)\} \tag{20}$$

$$\times\ \xi_3\{Kjj(\lambda')\}\xi_3(j^*Kj^*(\lambda))$$

Comparing equation 19 and equation 4 it is clear that

$$\alpha^j_{K(\varepsilon\delta)J(\beta\gamma)} = \chi\sqrt{n_J n_K}\ W\{jK^*J(\gamma\beta)j^*;j(\lambda)j^*(\lambda')\} \tag{21}$$

By using equations 9, 20, and 21 one can now evaluate the elements of the reordering matrices associated with the octet model whenever equation 12 is valid. Of course, all the reordering matrices associated with SU(3) may be evaluated using equation 10 even if instances are encountered where equation 12 is not valid. In this circumstance equation 21 would have to be abandoned but one could still simplify equation 10 for computational purposes in the same manner as the W-coefficients were simplified in the first section of this chapter. One would, of course, have to be careful to use a consistent phase convention. The result of this simplification using de Swart's phase convention for the octet model is

$$\alpha^{j}_{K(\varepsilon\delta)J(\beta\gamma)} = (-1)^{\overline{N}^{*}_{H}+\overline{M}^{*}_{H}} \left(\frac{n_J}{n_K}\right)^{\frac{1}{2}} \sum_{I_1 I_2 I_3} \sum_{Y'Y''} (-1)^{I_1+I_3+\frac{1}{2}(Y'-Y'')} \tag{22}$$

$$\times\ (2I_2+1)\delta(I_1,I_3,I_2)[(2I_1+1)(2I_3+1)]^{-\frac{1}{2}}$$

$$\times\ (j^{*}I_3Y-Y';jI_3Y'|J(\gamma)0Y)(j^{*}I_1Y''-Y;jI_3Y'|K''(\varepsilon)I_2Y'+Y''-Y)$$

$$\times\ (j^{*}I_3Y-Y';jI_1-Y''|K(\delta)I_2Y-Y''-Y')$$

$$\times\ (j^{*}I_1Y''-Y;jI_1-Y''|J^{*}(\beta)0-Y)$$

Equation 22 was actually used to evaluate the matrix elements of the SU(3) reordering matrix α^{8}. Recall that the IR's of SU(3) are labeled by their dimensions. Since (see refs. 23 and 56)

$$8 \times 8 = 1 + 8(1) + 8(2) + 10 + 10^{*} + 27 \tag{23}$$

there are eight scalars occurring in the decomposition of (8 x 8) x (8 x 8); one occurring in the decomposition of each of the following terms

$$1 \times 1, \quad 8(1) \times 8(1), \quad 8(1) \times 8(2), \quad 8(2) \times 8(1),$$

$$8(2) \times 8(2), \quad 10^{*} \times 10, \quad 10 \times 10^{*}, \text{ and } \quad 27 \times 27 \ .$$

This is a consequence of condition 4 following equation II-20 and the fact that for SU(3) an IR is equivalent to its contragredient if and only if its dimension is the cube of an integer (56). These scalers will be denoted respectively by

$$S^8_1 , \quad S^8_{8(1,1)} , \quad S^8_{8(1,2)} , \quad S^8_{8(2,1)}$$

$$S^8_{8(2,2)} , \quad S^8_{10} , \quad S^8_{10^*} , \quad \text{and} \quad S^8_{27} \ . \tag{24}$$

The corresponding reordering matrix α^8 with the rows and columns labeled in the same order as in the set 24 is given by Table 6.

It is worth noting that $\mathrm{Tr}\alpha^8 = 0$ so that in this case there are four P_{24} invariants with eigenvalue +1 and four with eigenvalue -1.

The real orthogonal matrix U given in Table 7 diagonalizes α^8. That is

$$U \, \alpha^8 \, U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25}$$

where 1 is the 4 x 4 unit matrix. Thus U can be used to construct the P_{24} invariants.

Biquadratic Scalars in the Eightfold Way

Using the notation for the scalars which was introduced just prior to equation II-18 and the C-G matrices defined by equation II-24 it

Table 6. The Reordering Matrix α^8

$K(\varepsilon\delta)/J(\beta\gamma)$	1	8(1,1)	8(1,2)	8(2,1)	8(2,2)	10	10^*	27
1	$1/8$	$-\sqrt{2}/4$	0	0	$-\sqrt{2}/4$	$\sqrt{10}/8$	$-\sqrt{10}/8$	$3\sqrt{3}/8$
8(1,1)	$-\sqrt{2}/4$	$-3/10$	0	0	$1/2$	$\sqrt{5}/5$	$-\sqrt{5}/5$	$-3\sqrt{6}/20$
8(1,2)	0	0	$-1/2$	$-1/2$	0	$1/2$	$1/2$	0
8(2,1)	0	0	$-1/2$	$-1/2$	0	$-1/2$	$-1/2$	0
8(2,2)	$-\sqrt{2}/4$	$1/2$	0	0	$1/2$	0	0	$\sqrt{6}/4$
10	$\sqrt{10}/8$	$\sqrt{5}/5$	$1/2$	$-1/2$	0	$1/4$	$-1/4$	$-\sqrt{30}/40$
10^*	$-\sqrt{10}/8$	$-\sqrt{5}/5$	$1/2$	$-1/2$	0	$-1/4$	$1/4$	$\sqrt{30}/40$
27	$3\sqrt{3}/8$	$-3\sqrt{6}/20$	0	0	$\sqrt{6}/4$	$-\sqrt{30}/40$	$\sqrt{30}/40$	$7/40$

Table 7. The Matrix U

$K(\varepsilon\delta)/J(\beta\gamma)$	1	8(1,1)	8(1,2)	8(2,1)	8(2,2)	10	10^*	27
1	0	0	$1/2$	$-1/2$	0	$1/2$	$1/2$	0
8(1,1)	$\sqrt{30}/8$	0	0	0	0	$\sqrt{3}/4$	$-\sqrt{3}/4$	$\sqrt{10}/8$
8(1,2)	$-\sqrt{14}/56$	0	0	0	$2\sqrt{7}/7$	$-\sqrt{35}/28$	$\sqrt{35}/28$	$5\sqrt{42}/56$
8(2,1)	$-\sqrt{70}/28$	$\sqrt{35}/10$	0	0	$\sqrt{35}/14$	$\sqrt{7}/7$	$-\sqrt{7}/7$	$-3\sqrt{210}/140$
8(2,2)	0	0	$\sqrt{6}/3$	0	0	$-\sqrt{6}/6$	$-\sqrt{6}/6$	0
10	0	0	$\sqrt{3}/6$	$\sqrt{3}/2$	0	$\sqrt{3}/6$	$\sqrt{3}/6$	0
10^*	$\sqrt{22}/11$	$5\sqrt{11}/22$	0	0	$-\sqrt{11}/22$	$-\sqrt{55}/22$	$\sqrt{55}/22$	0
27	$-3\sqrt{55}/44$	$3\sqrt{110}/110$	0	0	$-\sqrt{110}/22$	$\sqrt{22}/44$	$-\sqrt{22}/44$	$\sqrt{165}/20$

follows that

$$S^{j}_{J(\varepsilon\delta)}(\Psi\psi,\Phi\varphi) = \sum_{M} \Psi^{T} \left\langle jJ^{*}(\varepsilon)-M \right\rangle \psi \left\langle J\ 1\ 0 \right\rangle_{-MM} \times \Phi^{T} \left\langle jJ(\delta)M \right\rangle \varphi$$

$$= \sum_{II_{z}Y} \Psi(I_1 I_{1z} Y_1)(j^{*}I_1 Y_1; jI_2 Y_2|J^{*}(\varepsilon)I\text{-}Y) \left\langle\!\left\langle I_1 I_{1z}; I_2 I_{2z}|I\text{-}I_z \right\rangle\!\right\rangle \tag{26}$$

$$\times\ \psi(I_2 I_{2z} Y_2)(J^{*}I\text{-}Y; JIY|1\ 0\ 0) \left\langle\!\left\langle I\text{-}I_z; II_z|\ 0\ 0 \right\rangle\!\right\rangle$$

$$\times\ \Phi(I_3 I_{3z} Y_3)(j^{*}I_3 Y_3; jI_4 Y_4|J(\delta)IY) \left\langle\!\left\langle I_3 I_{3z}; I_4 I_{4z}|II_z \right\rangle\!\right\rangle$$

$$\times\ \varphi(I_4 I_{4z} Y_4)$$

where the isoscalar factors defined in equation 5 have been introduced and the sum is over all I, I_i, I_z, I_{iz}, Y and Y_i for $i = 1, 2, 3, 4$. Recombining the SU(2) C-G coefficients with the various states involved equation 26 can be written

$$S^{j}_{J(\varepsilon\delta)}(\Psi\psi,\Phi\varphi) = \sum_{IY} (j^{*}I_1 Y_1; jI_2 Y_2|J^{*}(\varepsilon)I\text{-}Y)(J^{*}I\text{-}Y; JIY|1\ 0\ 0)$$

$$\times\ (j^{*}I_3 Y_3; jI_4 Y_4|J(\delta)IY) S(I_1 I_2 I_3 I_4|IY_1 Y_2 Y_3 Y_4) \tag{27}$$

where $S(\ldots|\ldots)$ is an SU(2) scalar defined by

$$S(I_1I_2I_3I_4|IY_1Y_2Y_3Y_4) = \sum_{I_z} \Psi(I_1I_{1z}Y_1) \left\langle\!\left\langle I_1I_{1z};I_2I_{2z}|I-I_z \right\rangle\!\right\rangle \quad (28)$$

$$\times \psi(I_2I_{2z}Y_2) \left\langle\!\left\langle I-I_z;II_z|\ 0\ 0 \right\rangle\!\right\rangle \Phi(I_3I_{3z}Y_3)$$

$$\times \left\langle\!\left\langle I_3I_{3z};I_4I_{4z}|II_z \right\rangle\!\right\rangle \varphi(I_4I_{4z}Y_4)$$

According to de Swart's phase convention (equation 11)

$$\Psi(I_1I_{1z}Y_1) = (-1)^{I_{1z}+1/2\ Y_1} \left(\Psi'(I_1-I_{1z}-Y_1)\right)^* = (-1)^{I_{1z}+1/2\ Y_1}$$

$$\times \overline{\Psi'}(I_1I_{1z}Y_1) \qquad (29)$$

where Ψ' transforms according to the IRj and $\overline{\Psi'}$ is the complex conjugate transpone of Ψ' . A similar statement can be made for Φ . Now defining the matrices $(I_1Y_1I_2I)_{I_z}$ and $(I_1Y_1I_2I)^{I_z}$ by

$$\left((I_1Y_1I_2I)_{I_z}\right)_{I_{1z}I_{2z}} = (-1)^{I_{1z}+1/2\ Y_1} \left\langle\!\left\langle I_1I_{1z};I_2I_{2z}|II_z \right\rangle\!\right\rangle \qquad (30)$$

and

$$(I_1Y_1I_2I)^{I_z} = \left\langle\!\left\langle I - I_z;II_z|0\ 0 \right\rangle\!\right\rangle (I_1Y_1I_2I)_{-I_z} \qquad (31)$$

the SU(2) scalars can be written:

$$S(I_1I_2I_3I_4|IY_1Y_2Y_3Y_4) = \sum_{I_z} \overline{\Psi}(I_1Y_1)(I_1Y_1I_2I)^{I_z} \quad \psi(I_2Y_2) \qquad (32)$$

$$\times \varphi(I_3Y_3)(I_3Y_3I_4I)_{I_z}\varphi(I_4Y_4)$$

In equation 32 $\psi(I_2Y_2)$ is a column matrix whose elements are labeled by I_{2z} in increasing algebraic value, etc. $\bar{\Psi}(I_1Y_1)$ is thus a row matrix whose elements are labeled by I_{1z} in decreasing algebraic order. The subscripts and superscripts do not connote covariance and contravariance although it may be possible to redefine the matrices so that this is the case. The matrices of equations 30 and 31 can be easily evaluated by using tables of Clebsch-Gordan coefficients such as those of Condon and Shortley (63). In particular it should be noted that

$$\langle\langle I - I_z; II_z | \, 0 \; 0 \rangle\rangle = (-1)^{I+I_z}(2I+1)^{-1/2} \tag{33}$$

Furthermore the amount of work involved in evaluating these matrices can be considerably reduced if one observes from the definitions 30 and 31 that

$$(I_2Y_2I_1I)_{I_z} = (-1)^{I_1+I_z-I+I_z+1/2(Y_1+Y_2)}(I_1Y_1I_2I)^T_{I_z} \tag{34}$$

where T denotes the transpose across the antidiagonal. Making use of this result the matrices $(I_1Y_1I_2I)_{I_z}$ and $(I_1Y_1I_2I)^{I_z}$ have been evaluated and are tabulated in Table 8. The notation used in Table 8 is explained in Table 9. It is clear from Table 8 that s and S couple isospins 1/2 and 1 (in opposite orders) to obtain isospin 1/2, t and T couple 1/2 and 1 to obtain 3/2, and U couples 1 and 1 to obtain 2.

A check on the matrices can be made by using the result that

$$\mathrm{Tr}\left[(I_1Y_1I_2I)^T_{I_z}(I_1Y_1I_2I')^{-I'_z}\right] = \frac{(-1)^{I-I_z}}{\sqrt{2I+1}}\,\delta_{I_zI'_z}\,\delta_{II'} \tag{35}$$

Table 8. SU(2) Coupling Matrices

	$(I_1Y_1I_2I)^{I_z}$	$(I_1Y_1I_2I)_{I_z}$
1.	$(0000)^0 = 1$	$(0000)_0 = 1$
2.	$(00\ 1/2\ 1/2)^{1/2} = 1/\sqrt{2}\,(0\ -\ 1)$	$(00\ 1/2\ 1/2)_{1/2} = (10)$
	$(00\ 1/2\ 1/2)^{-1/2} = 1/\sqrt{2}\,(10)$	$(00\ 1/2\ 1/2)_{-1/2} = (01)$
3.	$(0011)^1 = 1/\sqrt{3}\,(001)$	$(0011)_1 = (100)$
	$(0011)^0 = 1/\sqrt{3}\,(0{-}10)$	$(0011)_0 = (010)$
	$(0011)^{-1} = 1/\sqrt{3}\,(100)$	$(0011)_{-1} = (001)$
4.	$(1/2\ -10\ 1/2)^{1/2} = 1/\sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}$	$(1/2\ -10\ 1/2)_{1/2} = \begin{pmatrix}0\\1\end{pmatrix}$
	$(1/2\ -10\ 1/2)^{-1/2} = 1/\sqrt{2}\begin{pmatrix}0\\1\end{pmatrix}$	$(1/2\ -10\ 1/2)_{-1/2} = \begin{pmatrix}-1\\0\end{pmatrix}$
5.	$(1/2\ 10\ 1/2)^{1/2} = 1/\sqrt{2}\begin{pmatrix}-1\\0\end{pmatrix}$	$(1/2\ 10\ 1/2)_{1/2} = \begin{pmatrix}0\\-1\end{pmatrix}$
	$(1/2\ 10\ 1/2)^{-1/2} = 1/\sqrt{2}\begin{pmatrix}0\\-1\end{pmatrix}$	$(1/2\ 10\ 1/2)_{-1/2} = \begin{pmatrix}1\\0\end{pmatrix}$

Table 8. SU(2) Coupling Matrices (Continued)

6.	$(1/2\ {-1}\ 1/2\ 0)^0 = 1/\sqrt{2}\ 1_2$	$(1/2\ {-1}\ 1/2\ 0)_0 = 1/\sqrt{2}\ 1_2$
7.	$(1/2\ 1\ 1/2\ 0)^0 = -1/\sqrt{2}\ 1_2$	$(1/2\ 1\ 1/2\ 0)_0 = -1/\sqrt{2}\ 1_2$
8.	$(1/2\ {-1}\ 1/2\ 1)^1 = -1/\sqrt{6}\ \tau_+$	$(1/2\ {-1}\ 1/2\ 1)_1 = 1/\sqrt{2}\ \tau_-$
	$(1/2\ {-1}\ 1/2\ 1)^0 = 1/\sqrt{6}\ \tau_0$	$(1/2\ {-1}\ 1/2\ 1)_0 = -1/\sqrt{2}\ \tau_0$
	$(1/2\ {-1}\ 1/2\ 1)^{-1} = 1/\sqrt{6}\ \tau_-$	$(1/2\ {-1}\ 1/2)_{-1} = -1/\sqrt{2}\ \tau_+$
9.	$(1/2\ 1\ 1/2\ 1)^1 = 1/\sqrt{6}\ \tau_+$	$(1/2\ 1\ 1/2\ 1)_1 = -1/\sqrt{2}\ \tau_-$
	$(1/2\ 1\ 1/2\ 1)^0 = -1/\sqrt{6}\ \tau_0$	$(1/2\ 1\ 1/2\ 1)_0 = 1/\sqrt{2}\ \tau_0$
	$(1/2\ 1\ 1/2\ 1)^{-1} = -\ 1/\sqrt{6}\ \tau_-$	$(1/2\ 1\ 1/2\ 1)_{-1} = 1/\sqrt{2}\ \tau_+$
10.	$(1/2\ {-11}\ 1/2)^\nu = -1/\sqrt{6}\ s^\nu$	$(1/2\ {-11}\ 1/2)_\nu = 1/\sqrt{3}\ s_\nu$
11.	$(1/2\ 11\ 1/2)^\nu = 1/\sqrt{6}\ s^\nu$	$(1/2\ 11\ 1/2)_\nu = -1/\sqrt{3}\ s_\nu$
12.	$(1/2\ {-11}\ 3/2)^\nu = -1/2\ t^\nu$	$(1/2\ {-11}\ 3/2)_\nu = t_\nu$

Table 8. SU(2) Coupling Matrices (Continued)

13.	$(1/2\ 11\ 3/2)^{\nu} = 1/2\ t^{\nu}$	$(1/2\ 11\ 3/2)_{\nu} = -t_{\nu}$
14.	$(1001)^{1} = 1/\sqrt{3}\begin{pmatrix}-1\\0\\0\end{pmatrix}$	$(1001)_{1} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$
	$(1001)^{0} = 1/\sqrt{3}\begin{pmatrix}0\\-1\\0\end{pmatrix}$	$(1001)_{0} = \begin{pmatrix}0\\1\\0\end{pmatrix}$
	$(1001)^{-1} = 1/\sqrt{3}\begin{pmatrix}0\\0\\-1\end{pmatrix}$	$(1001)_{-1} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$
15.	$(10\ 1/2\ 1/2)^{\nu} = -1/\sqrt{6}\ S^{\nu}$	$(10\ 1/2\ 1/2)_{\nu} = -1/\sqrt{3}\ S_{\nu}$
16.	$(10\ 1/2\ 3/2)^{\nu} = -1/2\ T^{\nu}$	$(10\ 1/2\ 3/2)_{\nu} = T_{\nu}$
17.	$(1010)^{0} = -1/\sqrt{3}\ 1_{3}$	$(1010)_{0} = -1/\sqrt{3}\ 1_{3}$
18.	$(1011)^{1} = 1/\sqrt{6}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$	$(1011)_{1} = 1/\sqrt{2}\begin{pmatrix}0&0&0\\-1&0&0\\0&-1&0\end{pmatrix}$

Table 8. SU(2) Coupling Matrices (Continued)

18. (Continued)

$$(1011)^{0} = 1/\sqrt{6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1011)_{0} = 1/\sqrt{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(1011)^{-1} = 1/\sqrt{6}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad (1011)_{-1} = 1/\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

19. $(1012)^{\nu} = 1/\sqrt{5}\, U^{\nu}$ $\qquad (1012)_{\nu} = U_{\nu}$

Table 9. Notation Used in Table 8

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1. 1_2 = 2 x 2 unit matrix $\qquad$ 1_3 = 3 x 3 unit matrix

2. $\tau_\pm = \frac{1}{\sqrt{2}}(\tau_1 \pm i\,\tau_2)$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. $$s_{1/2} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad s_{-1/2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$s^{1/2} = -\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \qquad s^{-1/2} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

4. $$t_{3/2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad t_{1/2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad t_{-1/2} = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad t_{-3/2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$t^{3/2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad t^{1/2} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad t^{-1/2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad t^{-3/2} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

5. $$s_{1/2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \qquad s_{-1/2} = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$s^{1/2} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad s^{-1/2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

Table 9. Notation Used in Table 8 (Continued)

6. $T_{3/2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$ $T_{1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$ $T_{-1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$ $T_{-3/2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$T^{3/2} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $T^{1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix}$ $T^{-1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \\ 0 & -1 \end{pmatrix}$ $T^{-3/2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$

7. $U_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ $U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ $U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$U_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $U_{-2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$U^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $U^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $U^0 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $U^{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

which is easily derived from the definitions 30 and 31

With the aid of Table 8 all possible biquadratic SU(2) scalars in the octet model can be written done. There are 86 such scalars. They are listed in Table 10. For convenience in writing down Table 10 the following correspondence has been made with the $1/2^+$ baryon octet:

$$p \sim | 1/2\ 1/2\ 1 \rangle$$

$$n \sim | 1/2\ -1/2\ 1 \rangle$$

$$-\Sigma^+ \sim | 1\ 1\ 0 \rangle$$

$$\Sigma^0 \sim | 1\ 0\ 0 \rangle$$

$$\Sigma^- \sim | 1\ -1\ 0 \rangle$$

$$\Lambda \sim | 0\ 0\ 0 \rangle$$

$$\Xi^0 \sim | 1/2\ 1/2\ -1 \rangle$$

$$\Xi^- \sim | 1/2\ -1/2\ -1 \rangle$$

Here ~ means "transforms like." Additional notation for Table 10 is in Table 11.

Finally, using equation 27, de Swart's tables of isoscalar factors and Table 10, the biquadratic SU(3) scalars can be written down. These are recorded in Table 12.

Table 10. SU(2) Biquadratic Scalars

$S(0\ 0\ 0\ 0 \mid 0\ 0\ 0\ 0\ 0)$	=	$\bar{\Lambda}\,\Lambda\,\bar{\Lambda}\,\Lambda$
$S(0\ 0\ 1/2\ 1/2 \mid 0\ 0\ 0\ -1\ 1)$	=	$1/\sqrt{2}\,\bar{\Lambda}\,\Lambda\,\bar{N}\,N$
$S(0\ 0\ 1/2\ 1/2 \mid 0\ 0\ 0\ 1\ -1)$	=	$-1/\sqrt{2}\,\bar{\Lambda}\,\Lambda\,\bar{\Xi}\,\Xi$
$S(0\ 0\ 1\ 1 \mid 0\ 0\ 0\ 0\ 0)$	=	$-1/\sqrt{3}\,\bar{\Lambda}\,\Lambda\,\bar{\Sigma}\cdot\Sigma$
$S(0\ 1/2\ 0\ 1/2 \mid 1/2\ 0\ -1\ 0\ 1)$	=	$-1/\sqrt{2}\,\bar{\Lambda}\,\Xi^{\nu}\,\bar{\Lambda}\,N_{\nu}$
$S(0\ 1/2\ 0\ 1/2 \mid 1/2\ 0\ 1\ 0\ -1)$	=	$-1/\sqrt{2}\,\bar{\Lambda}\,N^{\nu}\,\bar{\Lambda}\,\Xi_{\nu}$
$S(0\ 1/2\ 1/2\ 0 \mid 1/2\ 0\ -1\ 1\ 0)$	=	$-1/\sqrt{2}\,\bar{\Lambda}\,\Xi^{\nu}\,\bar{\Xi}_{\nu}\,\Lambda$
$S(0\ 1/2\ 1/2\ 0 \mid 1/2\ 0\ 1\ -1\ 0)$	=	$1/\sqrt{2}\,\bar{\Lambda}\,N^{\nu}\,\bar{N}_{\nu}\,\Lambda$
$S(0\ 1/2\ 1/2\ 1 \mid 1/2\ 0\ -1\ 1\ 0)$	=	$1/\sqrt{6}\,\bar{\Lambda}\,\Xi^{\nu}\,\bar{\Xi}s_{\nu}\,\Sigma$
$S(0\ 1/2\ 1/2\ 1 \mid 1/2\ 0\ 1\ -1\ 0)$	=	$-1/\sqrt{6}\,\bar{\Lambda}\,N^{\nu}\,\bar{N}s_{\nu}\,\Sigma$
$S(0\ 1/2\ 1\ 1/2 \mid 1/2\ 0\ -1\ 0\ 1)$	=	$1/\sqrt{6}\,\bar{\Lambda}\,\Xi^{\nu}\,\bar{\Sigma}S_{\nu}\,N$
$S(0\ 1/2\ 1\ 1/2 \mid 1/2\ 0\ 1\ 0\ -1)$	=	$1/\sqrt{6}\,\bar{\Lambda}\,N^{\nu}\,\bar{\Sigma}S_{\nu}\,\Xi$
$S(0\ 1\ 0\ 1 \mid 1\ 0\ 0\ 0\ 0)$	=	$-1/\sqrt{3}\,\bar{\Lambda}\,\Sigma\cdot\bar{\Lambda}\,\Sigma$
$S(0\ 1\ 1/2\ 1/2 \mid 1\ 0\ 0\ -1\ 1)$	=	$1/\sqrt{6}\,\bar{\Lambda}\,\Sigma\cdot\bar{N}\,\tau\,N$
$S(0\ 1\ 1/2\ 1/2 \mid 1\ 0\ 0\ 1\ -1)$	=	$-1/\sqrt{6}\,\bar{\Lambda}\,\Sigma\cdot\bar{\Xi}\,\tau\,\Xi$
$S(0\ 1\ 1\ 0 \mid 1\ 0\ 0\ 0\ 0)$	=	$-1/\sqrt{3}\,\bar{\Lambda}\,\Sigma\cdot\bar{\Sigma}\,\Lambda$
$S(0\ 1\ 1\ 1 \mid 1\ 0\ 0\ 0\ 0)$	=	$-i/\sqrt{6}\,\bar{\Lambda}\,\Sigma\cdot(\bar{\Sigma}\times\Sigma)$
$S(1/2\ 0\ 0\ 1/2 \mid 1/2\ -1\ 0\ 0\ 1)$	=	$1/\sqrt{2}\,\bar{N}^{\nu}\,\Lambda\,\bar{\Lambda}\,N_{\nu}$
$S(1/2\ 0\ 0\ 1/2 \mid 1/2\ 1\ 0\ 0\ -1)$	=	$-1/\sqrt{2}\,\bar{\Xi}^{\nu}\,\Lambda\,\bar{\Lambda}\,\Xi_{\nu}$
$S(1/2\ 0\ 1/2\ 0 \mid 1/2\ -1\ 0\ 1\ 0)$	=	$1/\sqrt{2}\,\bar{N}^{\nu}\,\Lambda\,\bar{\Xi}_{\nu}\,\Lambda$
$S(1/2\ 0\ 1/2\ 0 \mid 1/2\ 1\ 0\ -1\ 0)$	=	$1/\sqrt{2}\,\bar{\Xi}^{\nu}\,\Lambda\,\bar{N}_{\nu}\,\Lambda$

Table 10. SU(2) Biquadratic Scalars (Continued)

S(1/2 0 1/2 1|1/2 -1 0 1 0)	=	$-1/\sqrt{6}\ \bar{N}^{\nu}\Lambda\bar{\Xi}\ s_{\nu}\Sigma$
S(1/2 0 1/2 1|1/2 1 0 -1 0)	=	$-1/\sqrt{6}\ \bar{\Xi}^{\nu}\Lambda\bar{N}\ s_{\nu}\Sigma$
S(1/2 0 1 1/2|1/2 -1 0 0 1)	=	$-1/\sqrt{6}\ \bar{N}^{\nu}\Lambda\bar{\Sigma}\ S_{\nu}N$
S(1/2 0 1 1/2|1/2 1 0 0 -1)	=	$1/\sqrt{6}\ \bar{\Xi}^{\nu}\Lambda\bar{\Sigma}\ S_{\nu}\Xi$
S(1/2 1/2 0 0|0 -1 1 0 0)	=	$1/\sqrt{2}\ \bar{N}N\bar{\Lambda}\Lambda$
S(1/2 1/2 0 0|0 1 -1 0 0)	=	$-1/\sqrt{2}\ \bar{\Xi}\Xi\bar{\Lambda}\Lambda$
S(1/2 1/2 0 1|1 -1 1 0 0)	=	$1/\sqrt{6}\ \bar{N}\tau N\cdot\bar{\Lambda}\Sigma$
S(1/2 1/2 0 1|1 1 -1 0 0)	=	$-1/\sqrt{6}\ \bar{\Xi}\tau\Xi\cdot\Lambda\Sigma$
S(1/2 1/2 1/2 1/2|0 -1 -1 1 1)	=	$-1/2\ \bar{N}\Xi\bar{\Xi}N$
S(1/2 1/2 1/2 1/2|0 -1 1 -1 1)	=	$1/2\ \bar{N}N\bar{N}N$
S(1/2 1/2 1/2 1/2|0 -1 1 1 -1)	=	$-1/2\ \bar{N}N\bar{\Xi}\Xi$
S(1/2 1/2 1/2 1/2|0 1 -1 -1 1)	=	$-1/2\ \bar{\Xi}\Xi\bar{N}N$
S(1/2 1/2 1/2 1/2|0 1 -1 1 -1)	=	$1/2\ \bar{\Xi}\Xi\bar{\Xi}\Xi$
S(1/2 1/2 1/2 1/2|0 1 1 -1 -1)	=	$-1/2\ \bar{\Xi}N\bar{N}\Xi$
S(1/2 1/2 1/2 1/2|1 -1 -1 1 1)	=	$1/2\sqrt{3}\ \bar{N}\tau\Xi\cdot\bar{\Xi}\tau N$
S(1/2 1/2 1/2 1/2|1 -1 1 -1 1)	=	$-1/2\sqrt{3}\ \bar{N}\tau N\cdot\bar{N}\tau N$
S(1/2 1/2 1/2 1/2|1 -1 1 1 -1)	=	$1/2\sqrt{3}\ \bar{N}\tau N\cdot\bar{\Xi}\tau\Xi$
S(1/2 1/2 1/2 1/2|1 1 -1 -1 1)	=	$1/2\sqrt{3}\ \bar{\Xi}\tau\Xi\cdot\bar{N}\tau N$
S(1/2 1/2 1/2 1/2|1 1 -1 1 -1)	=	$-1/2\sqrt{3}\ \bar{\Xi}\tau\Xi\cdot\bar{\Xi}\tau\Xi$
S(1/2 1/2 1/2 1/2|1 1 1 -1 -1)	=	$1/2\sqrt{3}\ \bar{\Xi}\tau N\cdot\bar{N}\tau\Xi$
S(1/2 1/2 1 0 | 1 -1 1 0 0)	=	$1/\sqrt{6}\ \bar{N}\tau N\cdot\bar{\Sigma}\Lambda$
S(1/2 1/2 1 0 | 1 1 -1 0 0)	=	$-1/\sqrt{6}\ \bar{\Xi}\tau\Xi\cdot\bar{\Sigma}\Lambda$
S(1/2 1/2 1 1 | 0 -1 1 0 0)	=	$-1/\sqrt{6}\ \bar{N}N\bar{\Sigma}\cdot\Sigma$
S(1/2 1/2 1 1 | 0 1 -1 0 0)	=	$1/\sqrt{6}\ \bar{\Xi}\Xi\bar{\Sigma}\cdot\Sigma$

Table 10. SU(2) Biquadratic Scalars (Continued)

S(1/2 1/2 1 1 | 1 -1 1 0 0)	=	$i/2\sqrt{3}\ \bar{N}\tau N \cdot (\bar{\Sigma} \times \Sigma)$
S(1/2 1/2 1 1 | 1 1 -1 0 0)	=	$-i/2\sqrt{3}\ \bar{\Xi}\tau\Xi \cdot (\bar{\Sigma} \times \Sigma)$
S(1/2 1 0 1/2|1/2 -1 0 0 1)	=	$-1/\sqrt{6}\ \bar{N} s^{\nu} \Sigma \bar{\Lambda} N_{\nu}$
S(1/2 1 0 1/2|1/2 1 0 0 -1)	=	$1/\sqrt{6}\ \bar{\Xi} s^{\nu} \Sigma \bar{\Lambda} \Xi_{\nu}$
S(1/2 1 1/2 0|1/2 -1 0 1 0)	=	$-1/\sqrt{6}\ \bar{N} s^{\nu} \Sigma \bar{\Xi}_{\nu} \Lambda$
S(1/2 1 1/2 0|1/2 1 0 -1 0)	=	$-1/\sqrt{6}\ \bar{\Xi} s^{\nu} \Sigma \bar{N}_{\nu} \Lambda$
S(1/2 1 1/2 1|1/2 -1 0 1 0)	=	$1/3\sqrt{2}\ \bar{N} s^{\nu} \Sigma \bar{\Xi} s_{\nu} \Sigma$
S(1/2 1 1/2 1|1/2 1 0 -1 0)	=	$1/3\sqrt{2}\ \bar{\Xi} s^{\nu} \Sigma \bar{N} s_{\nu} \Sigma$
S(1/2 1 1/2 1|3/2 -1 0 1 0)	=	$1/2\ \bar{N} t^{\nu} \Sigma \bar{\Xi} t_{\nu} \Sigma$
S(1/2 1 1/2 1|3/2 1 0 -1 0)	=	$1/2\ \bar{\Xi} t^{\nu} \Sigma \bar{N} t_{\nu} \Sigma$
S(1/2 1 1 1/2|1/2 -1 0 0 1)	=	$1/3\sqrt{2}\ \bar{N} s^{\nu} \Sigma \bar{\Sigma} S_{\nu} N$
S(1/2 1 1 1/2|1/2 1 0 0 -1)	=	$-1/3\sqrt{2}\ \bar{\Xi} s^{\nu} \Sigma \bar{\Sigma} S_{\nu} \Xi$
S(1/2 1 1 1/2|3/2 -1 0 0 1)	=	$1/2\ \bar{N} t^{\nu} \Sigma \bar{\Sigma} T_{\nu} N$
S(1/2 1 1 1/2|3/2 1 0 0 -1)	=	$-1/2\ \bar{\Xi} t^{\nu} \Sigma \bar{\Sigma} T_{\nu} \Xi$
S(1 0 0 1 | 1 0 0 0 0)	=	$-1/\sqrt{3}\ \bar{\Sigma}\Lambda \cdot \bar{\Lambda}\Sigma$
S(1 0 1/2 1/2| 1 0 0 -1 1)	=	$1/\sqrt{6}\ \bar{\Sigma}\Lambda \cdot \bar{N}\tau N$
S(1 0 1/2 1/2| 1 0 0 1 -1)	=	$-1/\sqrt{6}\ \bar{\Sigma}\Lambda \cdot \bar{\Xi}\tau\Xi$
S(1 0 1 0 | 1 0 0 0 0)	=	$-1/\sqrt{3}\ \bar{\Sigma}\Lambda \cdot \bar{\Sigma}\Lambda$
S(1 0 1 1| 1 0 0 0 0)	=	$-i/\sqrt{6}\ \bar{\Sigma}\Lambda \cdot (\bar{\Sigma} \times \Sigma)$
S(1 1/2 0 1/2|1/2 0 -1 0 1)	=	$-1/\sqrt{6}\ \bar{\Sigma} S^{\nu} \Xi \bar{\Lambda} N_{\nu}$

Table 10. SU(2) Biquadratic Scalars (Continued)

S(1 1/2 0 1/2|1/2 0 1 0 -1)	=	$-1/\sqrt{6}\,\bar{\Sigma}\,S^{\nu}\,N\,\bar{\Lambda}\,\Xi_{\nu}$
S(1 1/2 1/2 0|1/2 0 -1 1 0)	=	$-1/\sqrt{6}\,\bar{\Sigma}\,S^{\nu}\,\Xi\,\bar{\Xi}_{\nu}\,\Lambda$
S(1 1/2 1/2 0|1/2 0 1 -1 0)	=	$1/\sqrt{6}\,\bar{\Sigma}\,S^{\nu}\,N\,\bar{N}_{\nu}\,\Lambda$
S(1 1/2 1/2 1|1/2 0 -1 1 0)	=	$1/3\sqrt{2}\,\bar{\Sigma}\,S^{\nu}\Xi\,\bar{\Xi}\,s_{\nu}\,\Sigma$
S(1 1/2 1/2 1|1/2 0 1 -1 0)	=	$-1/3\sqrt{2}\,\bar{\Sigma}\,S^{\nu}\,N\,\bar{N}\,s_{\nu}\,\Sigma$
S(1 1/2 1/2 1|3/2 0 -1 1 0)	=	$1/2\,\bar{\Sigma}\,T^{\nu}\,\Xi\,\bar{\Xi}t_{\nu}\,\Sigma$
S(1 1/2 1/2 1|3/2 0 1 -1 0)	=	$-1/2\,\bar{\Sigma}\,T^{\nu}\,N\,\bar{N}\,t_{\nu}\,\Sigma$
S(1 1/2 1 1/2|1/2 0 -1 0 1)	=	$1/3\sqrt{2}\,\bar{\Sigma}\,S^{\nu}\Xi\,\bar{\Sigma}\,S_{\nu}\,N$
S(1 1/2 1 1/2|1/2 0 1 0 -1)	=	$1/3\sqrt{2}\,\bar{\Sigma}\,S^{\nu}\,N\,\bar{\Sigma}\,S_{\nu}\,\Xi$
S(1 1/2 1 1/2|3/2 0 -1 0 1)	=	$1/2\,\bar{\Sigma}\,T^{\nu}\,\Xi\,\bar{\Sigma}\,T_{\nu}\,N$
S(1 1/2 1 1/2|3/2 0 1 0 -1)	=	$1/2\,\bar{\Sigma}\,T^{\nu}\,N\,\bar{\Sigma}\,T_{\nu}\,\Xi$
S(1 1 0 0 | 0 0 0 0 0)	=	$-1/\sqrt{3}\,\bar{\Sigma}\cdot\Sigma\,\bar{\Lambda}\,\Lambda$
S(1 1 0 1 | 1 0 0 0 0)	=	$-i/\sqrt{6}(\bar{\Sigma}\times\Sigma)\cdot\bar{\Lambda}\,\Sigma$
S(1 1 1/2 1/2|0 0 0 -1 1)	=	$-1/\sqrt{6}\,\bar{\Sigma}\cdot\Sigma\,\bar{N}\,N$
S(1 1 1/2 1/2|0 0 0 1 -1)	=	$1/\sqrt{6}\,\bar{\Sigma}\cdot\Sigma\,\bar{\Xi}\,\Xi$
S(1 1 1/2 1/2|1 0 0 -1 1)	=	$i/2\sqrt{3}\,(\bar{\Sigma}\times\Sigma)\cdot\bar{N}\,\tau\,N$
S(1 1 1/2 1/2|1 0 0 1 -1)	=	$-i/2\sqrt{3}\,(\bar{\Sigma}\times\Sigma)\cdot\bar{\Xi}\,\tau\,\Xi$
S(1 1 1 0 | 1 0 0 0 0)	=	$-i/\sqrt{6}\,(\bar{\Sigma}\times\Sigma)\cdot\bar{\Sigma}\,\Lambda$
S(1 1 1 1 | 0 0 0 0 0)	=	$1/3\,\bar{\Sigma}\cdot\Sigma\,\bar{\Sigma}\cdot\Sigma$
S(1 1 1 1 | 1 0 0 0 0)	=	$1/2\sqrt{3}\,(\bar{\Sigma}\times\Sigma)\cdot(\bar{\Sigma}\times\Sigma)$
S(1 1 1 1 | 2 0 0 0 0)	=	$1/\sqrt{5}\,\bar{\Sigma}\,U^{\nu}\,\Sigma\,\bar{\Sigma}\,U_{\nu}\,\Sigma$

Table 11. Notation for Table 10

1. $N = \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} N_{1/2} \\ N_{-1/2} \end{pmatrix} = \begin{pmatrix} -N^{-1/2} \\ N^{1/2} \end{pmatrix}$

$\bar{N} = (\bar{p}\ \bar{n}) = (\bar{N}^{1/2}\ \ \bar{N}^{-1/2}) = (\bar{N}_{-1/2}\ \ -\bar{N}_{1/2})$

2. $\Sigma = \begin{pmatrix} -\Sigma^{+} \\ \Sigma^{0} \\ \Sigma^{-} \end{pmatrix}$

$\Sigma \cdot \Sigma = \Sigma^{+}\,\Sigma^{-} + \Sigma^{0}\,\Sigma^{0} + \Sigma^{-}\,\Sigma^{+}$

$\bar{\Sigma} \cdot \Sigma = \overline{\Sigma^{+}}\,\Sigma^{+} + \overline{\Sigma^{0}}\,\Sigma^{0} + \overline{\Sigma^{-}}\,\Sigma^{-}$, etc.

Table 12. SU(3) Biquadratic Scalars

$$S_1^8 = 1/8(\bar\Xi\,\Xi + \bar N\,N + \bar\Sigma\cdot\Sigma + \bar\Lambda\,\Lambda)(\bar\Xi\,\Xi + \bar N\,N + \bar\Sigma\cdot\Sigma + \bar\Lambda\,\Lambda)$$

$$\begin{aligned}
S^8_{8(1,1)} = {} & \sqrt{2}/80(\sqrt{3}\,\bar\Xi\,s^\nu\,\Sigma + \sqrt{3}\,\bar\Sigma\,S^\nu\,N + \bar\Xi^\nu\,\Lambda + \bar\Lambda\,N^\nu) \\
& \times(-\sqrt{3}\,\bar N\,s_\nu\,\Sigma - \sqrt{3}\,\bar\Sigma\,S_\nu\,\Xi + \bar N_\nu\,\Lambda - \bar\Lambda\,\Xi_\nu) \\
& + \sqrt{2}/80(\sqrt{3}\,\bar N\,s^\nu\,\Sigma - \sqrt{3}\,\bar\Sigma\,S^\nu\,\Xi - \bar N^\nu\,\Lambda + \bar\Lambda\,\Xi^\nu) \\
& \times(\sqrt{3}\,\bar\Xi\,s_\nu\,\Sigma - \sqrt{3}\,\bar\Sigma\,S_\nu\,N + \bar\Xi_\nu\,\Lambda + \bar\Lambda\,N_\nu) \\
& + \sqrt{2}/80(-\sqrt{3}\,\bar\Xi\,\tau\,\Xi + \sqrt{3}\,\bar N\,\tau\,N + 2\,\bar\Lambda\,\Sigma + 2\,\bar\Sigma\,\Lambda) \\
& \cdot(\sqrt{3}\,\bar\Xi\,\tau\,\Xi - \sqrt{3}\,\bar N\,\tau\,N - 2\,\bar\Sigma\,\Lambda - 2\,\bar\Lambda\,\Sigma) \\
& + \sqrt{2}/80(\bar\Xi\,\Xi + \bar N\,N - 2\,\bar\Sigma\cdot\Sigma + 2\,\bar\Lambda\,\Lambda)(-\bar\Xi\,\Xi - \bar N\,N + 2\,\bar\Sigma\cdot\Sigma - 2\,\bar\Lambda\,\Lambda)
\end{aligned}$$

$$\begin{aligned}
S^8_{8(1,2)} = {} & \sqrt{30}/240(\sqrt{3}\,\bar\Xi\,s^\nu\,\Sigma + \sqrt{3}\,\bar\Sigma\,S^\nu\,N + \bar\Xi^\nu\,\Lambda + \bar\Lambda\,N^\nu) \\
& \times(\bar N\,s_\nu\,\Sigma - \bar\Sigma\,S_\nu\,\Xi + \sqrt{3}\,\bar N_\nu\,\Lambda + \sqrt{3}\,\bar\Lambda\,\Xi_\nu) \\
& + \sqrt{30}/240(\sqrt{3}\,\bar N\,s^\nu\,\Sigma - \sqrt{3}\,\bar\Sigma\,S^\nu\,\Xi - \bar N^\nu\,\Lambda + \bar\Lambda\,\Xi^\nu) \\
& \times(\bar\Xi\,s_\nu\,\Sigma + \bar\Sigma\,S_\nu\,N - \sqrt{3}\,\bar\Xi_\nu\,\Lambda + \sqrt{3}\,\bar\Lambda\,N_\nu) \\
& + \sqrt{30}/240(\sqrt{3}\,\bar\Xi\,\tau\,\Xi - \sqrt{3}\,\bar N\,\tau\,N - 2\,\bar\Sigma\,\Lambda - 2\,\bar\Lambda\,\Sigma) \\
& \cdot(\bar\Xi\,\tau\,\Xi + \bar N\,\tau\,N + 2i\,\bar\Sigma\times\Sigma) \\
& + \sqrt{10}/80(-\bar\Xi\,\Xi - \bar N\,N + 2\,\bar\Sigma\cdot\Sigma - 2\,\bar\Lambda\,\Lambda)(\bar\Xi\,\Xi - \bar N\,N)
\end{aligned}$$

Table 12. SU(3) Biquadratic Scalars (Continued)

$$S^{8}_{8(2,1)} = \sqrt{30}/240(\bar{\Xi}\, s^{\nu}\, \Sigma - \bar{\Sigma}\, S^{\nu}\, N - \sqrt{3}\, \bar{\Xi}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, N^{\nu})$$

$$\times\ (-\sqrt{3}\, \bar{N}\, s_{\nu}\, \Sigma - \sqrt{3}\, \bar{\Sigma}\, S_{\nu}\, \Xi + \bar{N}_{\nu}\, \Lambda - \bar{\Lambda}\, \Xi_{\nu})$$

$$+\ \sqrt{30}/240(\bar{N}\, s^{\nu}\, \Sigma + \bar{\Sigma}\, S^{\nu}\, \Xi + \sqrt{3}\, \bar{N}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, \Xi^{\nu})$$

$$\times\ (-\sqrt{3}\, \bar{\Xi}\, s_{\nu}\, \Sigma + \sqrt{3}\, \bar{\Sigma}\, S_{\nu}\, N - \bar{\Xi}_{\nu}\, \Lambda - \bar{\Lambda}\, N_{\nu})$$

$$+\ \sqrt{30}/240(\bar{\Xi}\, \tau\, \Xi + \bar{N}\, \tau\, N)\cdot(\sqrt{3}\, \bar{\Xi}\, \tau\, \Xi - \sqrt{3}\, \bar{N}\, \tau\, N - 2\, \bar{\Sigma}\Lambda - 2\, \bar{\Lambda}\Sigma)$$

$$-\ i\, \sqrt{30}/60(\bar{\Sigma} \times \Sigma)\cdot(\bar{\Sigma}\Lambda + \bar{\Lambda}\Sigma)$$

$$+\ \sqrt{10}/80(\bar{\Xi}\, \Xi - \bar{N}\, N)(-\bar{\Xi}\, \Xi - \bar{N}\, N + 2\, \bar{\Sigma} \cdot \Sigma - 2\, \bar{\Lambda}\, \Lambda)$$

$$+\ i\, \sqrt{10}/40\, \bar{\Sigma} \times \Sigma \cdot (\bar{\Xi}\, \tau\, \Xi - \bar{N}\, \tau\, N)$$

$$S^{8}_{8(2,2)} = \sqrt{2}/48(\bar{\Xi}\, s^{\nu}\, \Sigma - \bar{\Sigma}\, S^{\nu}\, N - \sqrt{3}\, \bar{\Xi}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, N^{\nu})$$

$$\times\ (\bar{N}\, s_{\nu}\, \Sigma - \bar{\Sigma}\, S_{\nu}\, \Xi + \sqrt{3}\, \bar{N}_{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, \Xi_{\nu})$$

$$+\ \sqrt{2}/48(\bar{N}\, s^{\nu}\, \Sigma + \bar{\Sigma}\, S^{\nu}\, \Xi + \sqrt{3}\, \bar{N}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, \Xi^{\nu})$$

$$\times\ (-\bar{\Xi}\, s_{\nu}\, \Sigma - \bar{\Sigma}\, S_{\nu}\, N + \sqrt{3}\, \bar{\Xi}_{\nu}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, N_{\nu})$$

$$-\ \sqrt{2}/48(\bar{\Xi}\, \tau\, \Xi + \bar{N}\, \tau\, N + 2\, i\, \bar{\Sigma} \times \Sigma)\cdot(\bar{\Xi}\, \tau\, \Xi + \bar{N}\, \tau\, N + 2\, i\, \bar{\Sigma} \times \Sigma)$$

$$-\ \sqrt{2}/16\, (\bar{\Xi}\Xi - \bar{N}\, N)\, (\bar{\Xi}\, \Xi - \bar{N}\, N)$$

Table 12. SU(3) Biquadratic Scalars (Continued)

$$S^{8}_{10} = \sqrt{10}/20(\bar{N}\, t^{\nu}\, \Sigma - \bar{\Sigma}\, T^{\nu}\, \Xi)(\bar{\Xi}\, t_{\nu}\, \Sigma - \bar{\Sigma}\, T_{\nu}\, N)$$

$$+ \sqrt{10}/120(\bar{\Xi}\, \tau\, \Xi + \bar{N}\, \tau\, N - i\, \bar{\Sigma} \times \Sigma + \sqrt{3}\, \bar{\Sigma}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, \Sigma)$$

$$\cdot(\bar{\Xi}\, \tau\, \Xi + \bar{N}\, \tau\, N - i\, \bar{\Sigma} \times \Sigma - \sqrt{3}\, \bar{\Sigma}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, \Sigma)$$

$$+ \sqrt{10}/120(-\bar{\Xi}\, s^{\nu}\, \Sigma + \bar{\Sigma}\, S^{\nu}\, N - \sqrt{3}\, \bar{\Xi}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, N^{\nu})$$

$$\times (\bar{N}\, s_{\nu}\, \Sigma - \bar{\Sigma}\, S_{\nu}\, \Xi - \sqrt{3}\, \bar{N}_{\nu}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, \Xi_{\nu})$$

$$+ \sqrt{10}/20\, \bar{\Xi}\, N\, \bar{N}\, \Xi$$

$$S^{8}_{10^*} = \sqrt{10}/20(\bar{\Xi}\, t^{\nu}\, \Sigma + \bar{\Sigma}\, T^{\nu}\, N)(\bar{N}\, t_{\nu}\, \Sigma + \bar{\Sigma}\, T_{\nu}\, \Xi)$$

$$+ \sqrt{10}/120(-\bar{N}\, \tau\, N - \bar{\Xi}\, \tau\, \Xi + i\, \bar{\Sigma} \times \Sigma + \sqrt{3}\, \bar{\Sigma}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, \Sigma)$$

$$\cdot(\bar{N}\, \tau\, N - \bar{\Xi}\, \tau\, \Xi - i\, \bar{\Sigma} \times \Sigma + \sqrt{3}\, \bar{\Sigma}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, \Sigma)$$

$$+ \sqrt{10}/120(-\bar{N}\, s^{\nu}\, \Sigma - \bar{\Sigma}\, S^{\nu}\, \Xi + \sqrt{3}\, \bar{N}^{\nu}\, \Lambda + \sqrt{3}\, \bar{\Lambda}\, \Xi^{\nu})$$

$$\times (\bar{\Xi}\, s_{\nu}\, \Sigma + \bar{\Sigma}\, S_{\nu} N + \sqrt{3}\, \bar{\Xi}_{\nu}\, \Lambda - \sqrt{3}\, \bar{\Lambda}\, N_{\nu})$$

$$- \sqrt{10}/20\, \bar{N}\, \Xi\, \bar{\Xi}\, N$$

Table 12. SU(3) Biquadratic Scalars (Continued)

$$S^{8}_{27} = -\sqrt{3}/18\,(\bar{\Xi}\,\tau\,N \cdot \bar{N}\,\tau\,\Xi + \bar{N}\,\tau\,\Xi \cdot \bar{\Xi}\,\tau\,N + 2\,\bar{\Sigma}\,U^{\nu}\,\Sigma\,\bar{\Sigma}\,U_{\nu}\,\Sigma)$$

$$-\sqrt{3}/18(\bar{\Xi}\,t^{\nu}\,\Sigma - \bar{\Sigma}\,T^{\nu}\,N)(\bar{N}\,t_{\nu}\,\Sigma - \bar{\Sigma}\,T_{\nu}N)$$

$$+\sqrt{3}/18(\bar{N}\,t^{\nu}\,\Sigma + \bar{\Sigma}\,T^{\nu}\,\Xi)(\bar{\Xi}\,t_{\nu}\,\Sigma + \bar{\Sigma}\,T_{\nu}\,N)$$

$$-\sqrt{3}/540(\bar{\Xi}\,s^{\nu}\,\Sigma + \bar{\Sigma}\,S^{\nu}\,N - 3\sqrt{3}\,\bar{\Xi}^{\nu}\,\Lambda - 3\sqrt{3}\,\bar{\Lambda}\,N^{\nu})$$

$$\times(\bar{N}\,s_{\nu}\,\Sigma + \bar{\Sigma}\,S_{\nu}\,\Xi + 3\sqrt{3}\,\bar{N}_{\nu}\,\Lambda - 3\sqrt{3}\,\bar{\Lambda}\,\Xi_{\nu})$$

$$-\sqrt{3}/90(\bar{\Xi}\,\tau\,\Xi - \bar{N}\,\tau\,N + \sqrt{3}\,\bar{\Sigma}\,\Lambda + \sqrt{3}\,\bar{\Lambda}\,\Sigma)$$

$$\cdot(\bar{\Xi}\,\tau\,\Xi - \bar{N}\,\tau\,N + \sqrt{3}\,\bar{\Sigma}\,\Lambda + \sqrt{3}\,\bar{\Lambda}\,\Sigma)$$

$$+\sqrt{3}/540(\bar{N}\,s^{\nu}\,\Sigma - \bar{\Sigma}\,S^{\nu}\,\Xi + 3\sqrt{3}\,\bar{N}^{\nu}\,\Lambda - 3\sqrt{3}\,\bar{\Lambda}\,\Xi^{\nu})$$

$$\times\,(\bar{\Xi}\,s_{\nu}\,\Sigma - \bar{\Sigma}\,S_{\nu}\,N - 3\sqrt{3}\,\bar{\Xi}_{\nu}\,\Lambda - 3\sqrt{3}\,\bar{\Lambda}\,N_{\nu})$$

$$-\sqrt{3}/1080(3\,\bar{\Xi}\,\Xi + 3\,\bar{N}\,N - \bar{\Sigma}\cdot\Sigma - 9\,\bar{\Lambda}\,\Lambda)(3\,\bar{\Xi}\,\Xi + 3\,\bar{N}\,N - \bar{\Sigma}\cdot\Sigma - 9\bar{\Lambda}\,\Lambda)$$

CHAPTER IV

A MODEL FOR WEAK INTERACTIONS

In this chapter an attempt will be made to write down a general weak interaction for leptonic decays of baryons which incorporates P_{13} invariance in the internal symmetry space in a non-trivial way. Such a model would automatically determine the relative strengths of strangeness nonconserving and strangeness conserving processes.

The $v - s$ invariant discussed in the first example of Chapter II serves as a good guide:

$$v - s = \overline{N}_1 \tau N_2 \cdot \overline{N}_3 \tau N_4 - \overline{N}_1 N_2 \overline{N}_3 N_4 .$$

When written out in detail $v - s$ becomes

$$v - s = 2(\overline{p}\, n\, \overline{e}\, \nu + \overline{n}\, p\, \overline{\nu}\, e - \overline{p}\, p\, \overline{e}\, e - \overline{n}\, n\, \overline{\nu}\, \nu) \qquad (1)$$

where the nucleon isodoublets N_3 and N_4 have been changed to "lepton isodoublets"; e.g., $N_3 = \begin{pmatrix} \nu \\ e \end{pmatrix}$. If the $V - A$ space-time structure is inserted,* the first two terms of $v - s$ are clearly the terms responsible for beta decay, electron capture and related processes. The last two terms are new and involve the so-called neutral or charge-retention currents.

It should be noted that as a weak interaction model equation 1 is

*The symbols p, n, e and ν represent either the wave functions or the field operators for the respective particles.

quite satisfactory for non-strange particle processes since it forbids elastic proton-neutrino and neutron-electron scattering but allows elastic proton-electron and neutron-neutrino scattering. The proton-neutrino scattering has been looked for but if present it occurs at a rate substantially less than the normal beta decay rate (see the Zichichi paper of ref. 28). On the other hand the weak proton-electron and neutron-neutrino scattering would be extremely difficult to detect for obvious reasons.

From a mathematical point of view v-s is an $SU(2)$ scalar and a P_{13} invariant with eigenvalue -1. Thus an obvious generalization to $SU(3)$ would be to select an $SU(3)$ biquadratic scalar which is P_{13} invariant with eigenvalue -1. One is met immediately with several obstacles.

(1) Such a scalar would conserve isospin and strangeness in nonleptonic processes which is known not to be the case in weak interaction. It is conceivable that one could define a "weak isospin" and a "weak strangeness" which are conserved in weak interactions and thus alleviate this problem. There is a precedent for this in the literature (64) which, however, is not suitable for the eightfold way.

(2) There appears to be no lepton octet analogous to the baryon octet. Such octets have been considered previously (65) but are rather unsatisfactory.

(3) From equation III-25 it is clear that there is not a unique P_{13} invariant with eigenvalue -1 in the eightfold way. On the contrary there are four independent invariants with this eigenvalue.

It is clear from these comments that some drastic assumptions are needed in order to carry through the generalization to $SU(3)$. Thus it will be assumed that:

(1) as far as the baryons are concerned the weak interaction is a P_{13} invariant SU(3) biquadratic scalar with eigenvalue -1

(2) the lepton current can replace any quadratic factor in the P_{13} invariant baryon expression which has commutation relations similar to those of the lepton current and its hermitian conjugate (see ref. 66), and

(3) the weak interaction is as simple as possible under assumptions 1 and 2.

This set of assumptions will be referred to as a schizon model for reasons which will soon be apparent.

From equation III-25 and Table 7 one can see by inspection that a "simple" P_{13} invariant SU(3) biquadratic scalar with eigenvalue -1 is obtained by taking the 8(2,2) row of U and adding to it $\sqrt{2}$ times the 10 row. The result is the quantity Q defined by

$$Q = \frac{\sqrt{3}}{2}\left[S^{8}_{8(1,2)} + S^{8}_{8(2,1)} \right] \qquad (2)$$

The uniqueness of Q should be especially noted here. It is the only possible P_{13} invariant with eigenvalue -1 having a total octet structure.

Now Q can be written more conveniently in terms of the F and D matrices of Gell-Mann.* In this notation

$$Q = \sum_{i} (\bar{N}\, F_i\, N\, \bar{N}\, D_i\, N + \bar{N}\, D_i\, N\, \bar{N}\, F_i\, N) \qquad (3)$$

*I. Ahrens has shown this explicitly in a private communication.

where an overall normalization factor has been discarded since the principle objective here is to determine relative strengths of different processes. Writing Q out explicitly gives

$$
\begin{aligned}
Q &= \overline{N}\,\frac{1}{\sqrt{2}}\,(F_1 - iF_2)N\,\overline{N}\,\frac{1}{\sqrt{2}}\,(D_1 + iD_2)N + \overline{N}\,\frac{1}{\sqrt{2}}\,(F_1 + iF_2)N\,\overline{N}\,\frac{1}{\sqrt{2}}\,(D_1 - iD_2)N + \ldots \\
&= \sqrt{2}\;\;\overline{p}n\left[\overline{N}\,\frac{1}{\sqrt{2}}\,(F_1 - iF_2)N + \overline{N}\,\frac{1}{\sqrt{2}}\,(D_1 - iD_2)N\right] \\
&\quad - \frac{1}{\sqrt{3}}\;\;\overline{p}\,\Lambda\left[\overline{N}\,\frac{1}{\sqrt{2}}\,(F_4 - iF_5)N + 3\,\overline{N}\,\frac{1}{\sqrt{2}}\,(D_4 - iD_5)N\right] \qquad (4) \\
&\quad + \frac{2}{\sqrt{3}}\,\overline{\Lambda}\,\Sigma^-\left[\overline{N}\,\frac{1}{\sqrt{2}}\,(F_1 - iF_2)N\right] \\
&\quad + \sqrt{2}\;\;\overline{n}\,\Sigma^-\left[\overline{N}\,\frac{1}{\sqrt{2}}\,(F_4 - iF_5)N - \overline{N}\,\frac{1}{\sqrt{2}}\,(D_4 - iD_5)N\right] + \ldots
\end{aligned}
$$

The expressions in the brackets are of the form $\overline{N}(F_j + aD_j)N$ where a takes on the values 1, 3, 0 and -1 respectively for the four brackets. These quantities have commutation relations of the form (see ref. 23)

$$[F_i, F_j + aD_j] = if_{ijk}(F_k + aD_k) \qquad (5)$$

In particular

$$[F_i, F_j + aD_j] = i\varepsilon_{ijk}(F_k + aD_k); \quad (i,j,k) = 1,2,3 \qquad (6)$$

where ε_{ijk} is the totally antisymmetric tensor of rank three with $\varepsilon_{123} = 1$.

Furthermore by setting

$$G_1 = F_4, \quad G_2 = F_5, \quad G_3 = \frac{1}{2}(F_3 + \sqrt{3}\,F_8)$$

and

$$H_1 = D_4, \quad H_2 = D_5, \quad H_3 = \frac{1}{2}(D_3 + \sqrt{3}\,D_8)$$

the following relations are obtained from equation 5:

$$[G_i, G_j + aH_j] = i\varepsilon_{ijk}(G_k + aH_k) \tag{7}$$

for arbitrary a.

Now it is known (66) that the leptonic weak current L_μ and its hermitian conjugate generate the group SU(2) just as the sets F_1, F_2, F_3 and G_1, G_2, G_3 do as can be seen from equation 6 and 7. Thus the quantities in the square brackets of equation 4 would have exactly the same commutation relations as L_μ and $L_\mu^\dagger$ except for the presence of the D_j and H_j. Assumption (2) of the schizon model only requires "similar" commutation relations. Equations 6 and 7 are sufficiently similar to the relations for L_μ and $L_\mu^\dagger$ that the square brackets in equation 4 can be replaced by the lepton current, L_μ:

$$L_\mu = \bar{e}\,\nu + \bar{\mu}\,\nu_\mu \tag{8}$$

It is clear from equation 4 that L_μ replaces a strangeness changing bracket in one case and a strangeness conserving bracket in another case. This is the origin of the name "schizon model," in analogy with the schizon model of intermediate bosons of Lee and Yang (47).

Thus the assumptions of a schizon model lead to the following expression for the baryon beta decay interaction

$$H_\beta = \frac{g}{\sqrt{2}}\left[\bar{p}\, n\, \bar{e}\, \nu - \frac{1}{\sqrt{6}}\,\bar{p}\, \Lambda\, \bar{e}\, \nu + \frac{2}{\sqrt{6}}\,\bar{\Lambda}\, \Sigma^-\, \bar{e}\, \nu + \bar{n}\, \Sigma^-\, \bar{e}\, \nu + \ldots\right] \quad (9)$$

Equation 9 has been normalized to give the well-known expression for the neutron decay. It is clear that equation 9 gives relations between strangeness conserving and strangeness nonconserving processes.

The real test of this model is, of course, a comparison of its predictions with experimental results. In order to make this comparison an expression for the branching ratios of baryon beta decays is derived in the next chapter.

CHAPTER V

BARYON BETA DECAYS

In this chapter a general expression for the branching ratios for decays of the type

$$n \rightarrow p + e^- + \bar{\nu} \tag{1}$$

and

$$\Lambda \rightarrow p + e^- + \bar{\nu} \tag{2}$$

will be determined by a perturbation calculation. A V-A interaction will be used since, as will be seen shortly, renormalization effects due to strong interactions result in a branching ratio expression which is sufficiently general to cover any V, A interaction after a reinterpretation of the coupling constants and form factors. Maximum parity violation and zero neutrino mass will also be assumed.

The basic interaction for the process

$$b_1 \rightarrow b_2 + e^- + \bar{\nu} \tag{3}$$

where b_1 and b_2 are baryons is thus given by

$$H = \frac{g}{\sqrt{2}}\Big[\bar{\psi}_2\,\gamma_\mu\,\psi_1\,\bar{\psi}_e\,\gamma_\mu(1+\gamma_5)\psi_\nu - \bar{\psi}_2(i\,\gamma_\mu\gamma_5)\psi_1\bar{\psi}_e(i\,\gamma_\mu\gamma_5)(1+\gamma_5)\psi_\nu\Big] + \text{h.c.}$$

After simplification this becomes

$$H = \frac{g}{\sqrt{2}} \bar{\psi}_2 \gamma_\mu (1+\gamma_5)\psi_1 \bar{\psi}_e \gamma_\mu (1+\gamma_5)\psi_2 + \text{h.c.} \tag{4}$$

In this expression ψ_1, ψ_2, ψ_e and ψ_ν are the field operators for b_1, b_2, e^- and ν respectively; g is the weak interaction coupling constant and h.c. denotes the hermitian conjugate of the first term. The γ's are the usual Dirac matrices, $\bar{\psi} = \psi^\dagger \gamma_4$ and $\psi^\dagger$ denotes the hermitian conjugate of ψ.

To first order in the coupling constant g the amplitude for the decay 3 is given by the S operator according to

$$M = \left\langle b_2 e^- \bar{\nu} \mid S \mid b_1 \right\rangle = -i \left\langle b_2 e^- \bar{\nu} \middle| \int H \, d^4x \middle| b_1 \right\rangle \tag{5}$$

where $|b_1\rangle$ is the initial physical state and $|b_2 e^- \bar{\nu}\rangle$ is the final physical state. Neglecting final state interactions so that plane wave states may be used M can be written in the usual fashion as

$$M = \frac{-ig}{\sqrt{2}\, V^2} \int e^{i(q_1 - q_2 - q_e - q_\nu)x} d^4x \, B_\mu \bar{u}_e^{(+)}(\vec{q}_e)\gamma_\mu (1+\gamma_5) u_\nu^{(-)}(-\vec{q}_\nu) \tag{6}$$

where q_1, q_2, q_e and q_ν are the four-momenta of b_1, b_2, e^- and $\bar{\nu}$ respectively; $u^{(+)}$ and $u^{(-)}$ denote the positive and negative energy Dirac plane wave eigenfunctions with spin idices suppressed; V is the normalization volume and B_μ denotes the baryon matrix element. A space-like metric will be used throughout this chapter. Thus, for example,

$$q_1 x = (q_1)_\mu x_\mu = \vec{q}_1 \cdot \vec{x} - (q_1)_o t \;, \quad \mu = 1,2,3,4 \tag{7}$$

with $\vec{q}_1$ and $(q_1)_o$ denoting the space and time components respectively of q_1, etc.

Now

$$\int e^{i(q_1-q_2-q_e-q_\nu)x}\, d^4x = (2\pi)^4\, \delta(q_1 - q_2 - q_e - q_\nu) \tag{8}$$

and formally

$$[\delta(q_1 - q_2 - q_e - q_\nu)]^2 = \frac{VT}{(2\pi)^4}\, \delta(q_1 - q_2 - q_e - q_\nu) \tag{9}$$

Thus the transition rate for the process 3 is

$$\frac{\delta W}{\delta T} = \frac{|M|^2}{T} = \frac{g^2(2\pi)^4}{2V^3}\, \delta(q_1 - q_2 - q_e - q_\nu) B_\mu B_\lambda^\dagger \tag{10}$$

$$\times\ \bar{u}_e^{(+)}(\vec{q}_e)\gamma_\mu(1+\gamma_5)u_\nu^{(-)}(-\vec{q}_\nu)\bar{u}_\nu^{(-)}(-\vec{q}_\nu)\gamma_4(1+\gamma_5)\gamma_\lambda\gamma_4 u_e^{(+)}(\vec{q}_e)$$

The partial decay width $\Gamma(b_1 \rightarrow b_2e^-\bar{\nu})$ is then obtained from equation 10 by integrating over phase space, summing on final spin states and averaging over initial spin states. Thus

$$\Gamma(b_1 \rightarrow b_2e^-\bar{\nu}) = \frac{1}{2}\sum_r \frac{V^3}{(2\pi)^9}\int d^3q_2 d^3q_e d^3q_\nu\, \frac{\delta W}{\delta t} \tag{11}$$

where the sum is over all spin indices r and the integration is over the spatial components of the momenta.

Lepton Spin Sums

The sums on the electron and neutrino spins, r_e and r_ν, can be performed using the spin sum rules in Appendix 1. The result is

$$\sum_{r_e}\sum_{r_\nu} \bar{u}_e^{(+)}(\vec{q}_e)\gamma_\mu(1+\gamma_5)u_\nu^{(-)}(-\vec{q}_\nu)\bar{u}_\nu^{(-)}(-\vec{q}_\nu)\gamma_4(1+\gamma_5)\gamma_\lambda\gamma_4 u_e^{(+)}(\vec{q}_e)$$

$$= \frac{-(-1)^{\delta_{\lambda 4}}}{4E_e E_\nu} \ \mathrm{Tr}\left[\gamma_\mu(1+\gamma_5)(i\gamma q_\nu)\gamma_\lambda(1+\gamma_5)(i\gamma q_e - m_e)\right] \qquad (12)$$

where Tr denotes the trace of the matrix and $E_e = \sqrt{\vec{q}_e^{\,2} + m_e^2}$, etc. For convenience in evaluating this and other traces some important rules for traces are tabulated in Table 13 (see ref. 67).

Letting $L_{\mu\lambda}$ denote the trace part of equation 12 and using Table 13 gives

$$L_{\mu\lambda} = \mathrm{Tr}(-2\gamma_\mu\gamma_\omega\gamma_\lambda\gamma_\rho + 2\gamma_\mu\gamma_\omega\gamma_\lambda\gamma_\rho\gamma_5)(q_\nu)_\omega\,(q_e)_\rho$$

$$= -8(\delta_{\mu\omega}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\omega\rho} + \delta_{\mu\rho}\delta_{\omega\lambda} - \varepsilon_{\mu\omega\lambda\rho})(q_\nu)_\omega(q_e)_\rho \qquad (13)$$

Using equations 10, 12 and 13 in equation 11 gives

$$\Gamma(b_1 \rightarrow b_2 e^- \bar{\nu}) = \frac{2g^2}{(2\pi)^5}\int d^3q_2 \int \frac{d^3q_e}{2E_e}\int \frac{d^3q_\nu}{2E_\nu}\,\delta(q_1-q_2-q_e-q_\nu)$$

$$\times \sum_{r_1}\sum_{r_2} B_\mu B_\lambda^\dagger\,(-1)^{\delta_{\lambda 4}} \qquad (14)$$

$$\times(\delta_{\mu\omega}\delta_{\lambda\rho} - \delta_{\mu\lambda}\delta_{\omega\rho} + \delta_{\mu\rho}\delta_{\omega\lambda} - \varepsilon_{\mu\omega\lambda\rho})(q_\nu)_\omega(q_e)_\rho$$

Lepton Momenta Integration

The integration on the lepton momenta can now be performed exactly.*

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*Several parts of this chapter follow the analyses for similar processes in Källén's book (50).

Table 13. Rules for Evaluating Traces

$$\mathrm{Tr}(A + B) = \mathrm{Tr}A + \mathrm{Tr}B$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

Tr (odd number of γ_μ x any number of γ_5) $= 0$, $\mu = 1,2,3,4$

Tr (less than four γ_μ x odd number of γ_5) $= 0$, $\mu = 1,2,3,4$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\delta_{\mu\nu}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) = 4\varepsilon_{\mu\nu\rho\sigma}$$

$\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric

tensor of rank four with $\varepsilon_{1234} = 1$

$$\mathrm{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}) = \sum_{i \neq 1} (-1)^i \, \delta_{\mu_1 \mu_i} \, \mathrm{Tr}(\gamma_{\mu_2} \cdots \gamma_{\mu_{i-1}} \gamma_{\mu_{i+1}} \cdots \gamma_{\mu_n})$$

$$I_{\omega\rho} = \int \frac{d^3q_e}{2E_e} \int \frac{d^3q_\nu}{2E_\nu} \delta(q_1 - q_2 - q_e - q_\nu)(q_\nu)_\omega (q_e)_\rho \tag{15}$$

The integration can be expressed in a manifestly covariant form by using the θ-function defined by

$$\begin{aligned} \theta(q) = \theta(q_o) &= 1 \quad \text{if} \quad q_o > 0 \\ &= 0 \quad \text{if} \quad q_o < 0 \end{aligned} \tag{16}$$

and by noting that

$$\begin{aligned} \delta(q^2 + m^2)\theta(q) &= \delta(q_o^2 - (\vec{q}^2 + m^2))\theta(q) \\ &= \frac{1}{2\sqrt{\vec{q}^2 + m^2}} \delta(q_o - \sqrt{\vec{q}^2 + m^2}) \\ &= \frac{1}{2E} \delta(q_o - E) \end{aligned} \tag{17}$$

Thus equation 15 can be written

$$I_{\omega\rho} = \int d^4q_e\, d^4q_\nu\, \delta(q_e^2 + m_e^2)\delta(q_\nu^2)\theta(q_e)\theta(q_\nu)\delta(q_1 - q_2 - q_e - q_\nu)(q_\nu)_\omega (q_e)_\rho \tag{18}$$

which is manifestly Lorentz covariant. From Lorentz covariance it is clear that

$$I_{\omega\rho} = \alpha\delta_{\omega\rho} + \beta Q_\omega Q_\rho \tag{19}$$

where $Q = q_1 - q_2$.

Thus

$$I_{\rho\rho} = 4\alpha + \beta Q^2 = \int dq_e \int dq_\nu \delta(Q - q_\nu - q_e)\theta(q_e)\theta(q_\nu)\delta(q_e^2 + m_e^2)\delta(q_\nu^2) q_e q_\nu \tag{20}$$

But the δ-functions require that

$$Q = q_\nu + q_e$$

so that

$$Q^2 = q_\nu^{\,2} + q_e^{\,2} + 2q_\nu q_e = -m_e^{\,2} + 2q_\nu q_e$$

or

$$q_\nu q_e = \frac{1}{2}\,(Q^2 + m_e^{\,2})$$

Hence

$$I_{o\rho} = \frac{1}{2}(Q^2 + m_e^{\,2}) \int\int dq_e dq_\nu \delta(Q - q_\nu - q_e)\theta(q_e)\theta(q_\nu)\delta(q_e^{\,2} + m_e^{\,2})\delta(q_\nu^{\,2}) \quad (21)$$

Note that Q is the sum of a time-like vector and a null vector with positive time components and hence must be a time-like vector with a positive time component. Thus since $I_{o\rho}$ is invariant it can be evaluated in a coordinate system in which $\vec{Q} = 0$ without loss of generality.

Thus

$$I_{o\rho} = \frac{1}{2}(Q^2 + m_e^{\,2}) \int dq_\nu\ \theta(q_\nu)\theta(Q - q_\nu)\delta(q_\nu^{\,2})\,\delta((Q - q_\nu)^2 + m_e^{\,2})$$

$$= \frac{1}{2}\,(Q^2 + m_e^{\,2}) \int dq_\nu\ \theta(q_\nu)\theta(Q - q_\nu)\delta(q_\nu^{\,2})\,\delta(Q^2 + m_e^{\,2} + 2Q_o q_{\nu o})$$

$$= \frac{Q^2 + m_e^{\,2}}{4Q_o} \int d\Omega_\nu \int |\vec{q}_\nu|^2 d|\vec{q}_\nu| \int dq_{\nu o}\theta(q_\nu)\theta(Q - q_\nu)\delta(q_\nu^2)\delta\left(q_{\nu o} + \frac{Q^2 + m_e^{\,2}}{2Q_o}\right)$$

$$= \frac{\pi(Q^2 + m_e^{\,2})}{Q_o} \int |\vec{q}_\nu|^2\ d|\vec{q}_\nu|\ \theta(Q_o - E_\nu)\ \theta(E_\nu)\ \delta(\vec{q}_\nu^{\,2} - E_\nu^{\,2})$$

with $E_\nu = \dfrac{-(Q^2 + m_e^2)}{2Q_o} = \dfrac{Q_o^2 - m_e^2}{2Q_o}$

Then

$$I_{\rho\rho} = \frac{\pi(Q^2 + m_e^2)}{Q_o} \int |\vec{q_\nu}|^2 \, d|\vec{q_\nu}| \; \theta(Q_o - E_\nu)\theta(E_\nu) \, \frac{1}{2E_\nu} \; \delta(|\vec{q_\nu}| - E_\nu)$$

$$= \frac{\pi(Q^2 + m_e^2)}{2Q_o} \, E_\nu \; \theta(Q_o - E_\nu) \; \theta(E_\nu)$$

$$= \frac{\pi}{2} \, (Q^2 + m_e^2) \; \frac{Q_o^2 - m_e^2}{2Q_o^2} \quad \theta(Q - q_\nu)\theta(q_\nu); \; q_{\nu o} = E_\nu$$

$$= \frac{\pi}{2} \, (Q^2 + m_e^2) \; \frac{(Q^2 + m_e^2)}{2Q^2} \, \theta(Q - q_\nu) \; \theta(q_\nu)$$

$$= \frac{\pi(Q^2 + m_e^2)^2}{4Q^2} \quad \theta(Q - q_\nu) \; \theta(q_\nu) \tag{22}$$

Now $\theta(Q - q_\nu)$ requires $Q_o - \dfrac{Q_o^2 - m_e^2}{2Q_o} > 0$. Furthermore $Q_o > 0$. Thus it follows that

$$2Q_o^2 - Q_o^2 + m_e^2 > 0$$

or

$$Q_o^2 + m_e^2 > 0 \tag{23}$$

Hence $-Q^2 + m_e^2 > 0$.

Similarly $\theta(q_\nu)$ requires $E_\nu > 0$

or $$-(Q^2 + m_e^2) > 0 \tag{24}$$

The inequality 24 is more restrictive than 23 so that it includes the latter.

Thus

$$I_{\rho\rho} = \frac{\pi(Q^2 + m_e^2)^2}{4Q^2} \theta(-Q^2 - m_e^2)\theta(Q) \tag{25}$$

Hence

$$4\alpha + \beta Q^2 = \frac{\pi(Q^2 + m_e^2)^2}{4Q^2} \theta(-Q^2 - m_e^2)\theta(Q) \tag{26}$$

To get another relation between α and β consider

$$I_{\omega\rho}Q_\omega Q_\rho = \int dq_e dq_\nu \, \delta(Q-q_\nu-q_e)\theta(q_e)\theta(q_\nu)\delta(q_e^2+m_e^2)\delta(q_\nu^2)\, q_e Q q_\nu Q \tag{27}$$

Now $Qq_e = \frac{1}{2}\left[Q^2 + q_e^2 - (Q - q_e)^2\right]$

$$= \frac{1}{2}\left[Q^2 + q_e^2 - q_\nu^2\right] = \frac{1}{2}(Q^2 - m_e^2) \tag{28}$$

and similarly

$$Qq_\nu = \frac{1}{2}(Q^2 + m_e^2) \tag{29}$$

Hence

$$I_{\omega\rho}Q_\omega Q_\rho = \frac{1}{4}(Q^4 - m_e^4) \int dq_\nu \, \theta(Q - q_\nu)\theta(q_\nu)\delta((Q - q_\nu)^2 + m_e^2)\delta(q_\nu^2) \tag{30}$$

Again consider the frame in which $\vec{Q} = 0$. In this frame

$$I_{\omega\rho} Q_\omega Q_\rho = \frac{(Q^4 - m_e^4)}{8Q_o} 4\pi \int |\vec{q_\nu}|^2 \, d|\vec{q_\nu}| \theta(Q - q_\nu) \theta(Q_\nu) \delta(q_\nu^2) \tag{31}$$

with $E_\nu = -\dfrac{(Q^2 + m_e^2)}{2Q_o} = q_{\nu o}$. Now $\delta(q_\nu^2) = \delta(|\vec{q_\nu}|^2 - E_\nu^2)$.

Thus

$$I_{\omega\rho} Q_\omega Q_\rho = \frac{\pi(Q^2 - m_e^2)(Q^2 + m_e^2)^2}{8Q^2} \theta(-Q^2 - m_e^2) \theta(Q) \tag{32}$$

Also

$$I_{\omega\rho} Q_\omega Q_\rho = (\alpha\delta_{\omega\rho} + \beta Q_\omega Q_\rho) Q_\omega Q_\rho = \alpha Q^2 + \beta Q^4 \tag{33}$$

Letting

$$\varphi = \frac{\pi(Q^2 + m_e^2)^2}{8Q^2} \theta(-Q^2 - m_e^2) \theta(Q) \tag{34}$$

equations 26, 32 and 33 become

$$4\alpha + \beta Q^2 = 2\varphi \tag{35}$$

$$\alpha Q^2 + \beta Q^4 = (Q^2 - m_e^2)\varphi \tag{36}$$

Solving for α and β gives

$$\beta = \frac{2}{3Q^2} \left(1 - \frac{2m_e^2}{Q^2} \right) \varphi \tag{37}$$

and

$$\alpha = \frac{1}{3}\left(1 + \frac{m_e^2}{Q^2}\right)\varphi \tag{38}$$

Hence

$$I_{\omega\rho} = \frac{1}{3}\left(1 + \frac{m_e^2}{Q^2}\right)\varphi\,\delta_{\omega\rho} + \frac{2}{3Q^2}\left(1 - \frac{2m_e^2}{Q^2}\right)Q_\omega Q_\rho\,\varphi$$

$$= \frac{\pi}{24}\left(1 + \frac{m_e^2}{Q^2}\right)^2\left[\delta_{\omega\rho}Q^2\left(1 + \frac{m_e^2}{Q^2}\right) + 2Q_\omega Q_\rho\left(1 - \frac{2m_e^2}{Q^2}\right)\right]$$

$$\times\ \theta(-Q^2 - m_e^2)\,\theta(Q) \tag{39}$$

It is immediately clear from equation 39 that $I_{\omega\rho}$ is symmetric, i.e. $I_{\omega\rho} = I_{\rho\omega}$. Thus when $I_{\omega\rho}$ is contracted with $\varepsilon_{\mu\omega\lambda\rho}$ the result is zero. Equation 14 then becomes

$$\Gamma(b_1 \rightarrow b_2 e^- \bar{\nu}) = \frac{2g^2}{(2\pi)^5}\int d^3q_2 \sum_{r_1 r_2} B_\mu B_\lambda^\dagger\,(-1)^{\delta_{\lambda 4}}(2I_{\mu\lambda} - I_{\omega\omega}\delta_{\mu\lambda}) \tag{40}$$

Baryon Contribution

In the absence of strong interactions B_μ would be given by

$$B_\mu = \bar{u}_2^{(+)}(\vec{q}_2)\,\gamma_\mu(1 + \gamma_5)u_1^{(+)}(\vec{q}_1) \tag{41}$$

However, strong interactions between the baryons and mesons "renormalize" this matrix element in an essentially unknown fashion. Fortunately the most general form for B_μ including strong interactions effects can be written down from Lorentz invariance. Since B_μ must be a linear combination of vector and axial vector terms constructed from γ_μ, γ_5, q_1 and q_2, it can contain any of the partial list of candidates in Table 14.

Table 14. Induced Vector and Axial Vector Terms

Vectors	Axial Vectors
γ_μ	$\gamma_\mu\gamma_5$
$Q_\mu = (q_1 - q_2)_\mu$	$Q_\mu\gamma_5$
$\sigma_{\mu\nu}Q_\nu$, $\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$	$\sigma_{\mu\nu}Q_\nu\gamma_5$
$(q_1 + q_2)_\mu$	$(q_1 + q_2)_\mu\gamma_5$
$\sigma_{\mu\nu}(q_1 + q_2)_\nu$	$\sigma_{\mu\nu}(q_1 + q_2)_\nu\gamma_5$

There are possibilities other than those listed in Table 14. However, this list is already redundant since among all such possibilities only the first three in each list are independent (68). Thus the most general form for B_μ is given by

$$B_\mu = \bar{u}_2^{(+)}(\vec{q}_2)\Big[F_1(Q^2)\gamma_\mu + F_2(Q^2)\sigma_{\mu\nu}Q_\nu - iF_3(Q^2)Q_\mu + G_1(Q^2)\gamma_\mu\gamma_5 - iG_2(Q^2)Q_\mu\gamma_5 + G_3(Q^2)\sigma_{\mu\nu}Q_\nu\gamma_5\Big]u_1^{(+)}(\vec{q}_1) \qquad (42)$$

where the F's and G's are functions of Q^2 only and the imaginary units i have been inserted for convenience. The terms involving F_1, F_2, G_1 and G_2 are commonly referred to as first class amplitudes while those involving F_3 and G_3 are referred to as second class amplitudes (69). The second class amplitudes are zero if one assumes either full symmetry of the hadrons under SU(3) (69,70) or G-conjugation invariance (68) of B_ν. In fact for the process 1 the nucleon part of equation 4 is known to be G-conjugation invariant. Thus the invariance of the strong interaction under G-conjugation automatically requires G-conjugation invariance of the B_μ for that process. Unfortunately such a statement cannot be made in general.

In any event the functions F_2, F_3, G_2 and G_3 are not known except empirically for the nucleon, and will be omitted in the following. Furthermore F_1 and G_1 will be assumed constant and equal to their zero momentum transfer values. This assumption appears to be valid to within 2% up to momentum transfers of about 120 mev (see the Zichichi paper of ref. 28). Thus*

$$B_\mu = \bar{u}_2^{(+)}(\vec{q}_2)(F_1\gamma_\mu + G_1\gamma_\mu\gamma_5)u_1^{(+)}(\vec{q}_1) \tag{43}$$

Then

$$(-1)^{\delta_{\lambda 4}} B_\lambda^\dagger = -\bar{u}_1^{(+)}(\vec{q}_1)(F_1^*\gamma_\lambda + G_1^*\gamma_\lambda\gamma_5)u_2^{(+)}(\vec{q}_2) \tag{44}$$

where the asterisk denotes complex conjugation. Hence

$$\sum_{r_1 r_2} (-1)^{\delta_{\lambda 4}} B_\mu B_\lambda^\dagger = -\sum_{r_1 r_2} \bar{u}_2^{(+)}(\vec{q}_2)(F_1\gamma_\mu + G_1\gamma_\mu\gamma_5)u_1^{(+)}(\vec{q}_1)$$
$$\times\, \bar{u}_1^{(+)}(\vec{q}_1)(F_1^*\gamma_\lambda + G_1^*\gamma_\lambda\gamma_5)u_2^{(+)}(\vec{q}_2)$$

---

*Obviously equation 43 can accommodate any V, A interaction by reinterpreting the form factors.

$$= - \frac{1}{4E_1E_2} \operatorname{Tr} \Big[(F_1\gamma_\mu + G_1\gamma_\mu\gamma_5)(i\gamma q_1 - m_1)$$

$$\times (F_1^*\gamma_\lambda + G_1^*\gamma_\lambda\gamma_5)(i\gamma q_2 - m_2) \Big] \tag{45}$$

where the second equality follows as before from the results of Appendix 1. Equation 45 can be written in the following form

$$\sum_{r_1} \sum_{r_2} B_\mu B_\lambda^\dagger (-1)^{\delta_{\lambda 4}} = - \frac{1}{4E_1E_2} \Big[|F_1|^2 R_{\mu\lambda}^{(V,V)} - i F_1 G_1^* R_{\mu\lambda}^{(V,A)}$$

$$-i G_1 F_1^* R_{\mu\lambda}^{(A,V)} - |G_1|^2 R_{\mu\lambda}^{(A,A)} \Big] \tag{46}$$

where

$$R^{(i,j)} = \operatorname{Tr} \Big[O_i (i\gamma q_1 - m_1) O_j (i\gamma q_2 - m_2) \Big] \tag{47}$$

The indices i and j denote any of the five Lorentz covariants: scalar (S), vector (V), tensor (T), axial vector (A) or pseudoscalar (P) while O_i and O_j are any of the components of these covariants. The general procedure for evaluating the $R^{(i,j)}$ is given in Appendix 2. The results are tabulated in Table 15. Although the results for $R^{(i,j)}$ when the scalar, tensor or pseudoscalar covariants appear are not needed in this discussion, they are included in Table 15 for completeness.

From Table 15 it is seen that $R_{\mu\lambda}^{(V,A)}$ and $R_{\mu\lambda}^{(A,V)}$ are antisymmetric in μ and λ so that their contractions with $2I_{\mu\lambda} - I_{\omega\omega}\delta_{\mu\lambda}$ are zero. Hence equation 40 becomes

Table 15. $R^{(i,j)} = \mathrm{Tr}\left[O_i(i\gamma q_1 - m_1)O_j(i\gamma q_2 - m_2)\right]$

$i \backslash j$		S	V	T	A	P
	$O_i \backslash O_j$	1	γ_ω	$\sigma_{\omega\rho}$	$i\gamma_\omega\gamma_5$	γ_5
S	1	$4(m_1m_2-q_1q_2)$	$-4i\left[m_2(q_1)_\omega + m_1(q_2)_\omega\right]$	$4i\left[(q_1)_\omega(q_2)_\rho-(q_1)_\rho(q_2)_\omega\right]$	0	0
V	γ_μ	$-4i\left[m_2(q_1)_\mu + m_1(q_2)_\mu\right]$	$4\delta_{\mu\omega}(m_1m_2+q_1q_2)$ $-4\left[(q_1)_\mu(q_2)_\omega + (q_1)_\omega(q_2)_\mu\right]$	$-4\delta_{\mu\rho}(m_2q_1-m_1q_2)_\omega$ $+4\delta_{\mu\omega}(m_2q_1-m_1q_2)_\rho$	$-4i\varepsilon_{\mu\omega\alpha\beta}(q_1)_\alpha(q_2)_\beta$	0
T	$\sigma_{\mu\nu}$	$4i\left[(q_2)_\mu(q_1)_\nu -(q_1)_\nu(q_2)_\mu\right]$	$-4\delta_{\omega\nu}(m_1q_2-m_2q_1)_\mu$ $+4\delta_{\mu\omega}(m_1q_2-m_2q_1)_\nu$	$4(\delta_{\mu\omega}\delta_{\nu\rho}-\delta_{\nu\omega}\delta_{\mu\rho})(m_1m_2-q_1q_2)$ $+4\delta_{\nu\rho}\left[(q_1)_\mu(q_2)_\omega+(q_1)_\omega(q_2)_\mu\right]$ $+4\delta_{\mu\omega}\left[(q_1)_\nu(q_2)_\rho+(q_1)_\rho(q_2)_\nu\right]$ $-4\delta_{\nu\omega}\left[(q_1)_\mu(q_2)_\rho+(q_1)_\rho(q_2)_\mu\right]$ $-4\delta_{\mu\rho}\left[(q_1)_\nu(q_2)_\omega+(q_1)_\omega(q_2)_\nu\right]$	$4i\left[m_1(q_2)_\alpha + m_2(q_1)_\alpha\right]\varepsilon_{\mu\nu\omega\alpha}$	$4i(q_1)_\alpha(q_2)_\beta\,\varepsilon_{\mu\nu\alpha\beta}$
A	$i\gamma_\mu\gamma_5$	0	$-4i\varepsilon_{\omega\mu\alpha\beta}(q_2)_\alpha(q_1)_\beta$	$4i\left[m_2(q_1)_\alpha+m_1(q_2)_\alpha\right]\varepsilon_{\omega\rho\mu\alpha}$	$-4\delta_{\mu\omega}(q_1q_2-m_1m_2)$ $+4\left[(q_1)_\mu(q_2)_\omega + (q_1)_\omega(q_2)_\mu\right]$	$-4\left[m_2(q_1)_\mu-m_1(q_2)_\mu\right]$
P	γ_5	0	0	$4i(q_2)_\alpha(q_1)_\beta\,\varepsilon_{\omega\rho\alpha\beta}$	$-4\left[m_1(q_2)_\omega-m_2(q_1)_\omega\right]$	$4(m_1m_2+q_1q_2)$

$$\Gamma(b_1 \rightarrow b_2\ e^-\ \bar{\nu}) = -\frac{2g^2}{(2\ \pi)^5 2E_1} \int \frac{d^3q_2}{2E_2} \tag{48}$$

$$\times \left[|F_1|^2\ R_{\mu\lambda}^{(V,V)} - |G_1|^2\ R_{\mu\lambda}^{(A,A)} \right]$$

$$\times\ (2I_{\mu\lambda} - I_{\omega\omega}\ \delta_{\mu\lambda})$$

$$= \frac{g^2}{(2\ \pi)^5\ E_1} \int dq_2\ \delta(q_2^2 + m_2^2)\Theta(q_2)$$

$$\times \left[|F_1|^2 \left(-2R_{\mu\lambda}^{(V,V)}\ I_{\mu\lambda} + I_{\omega\omega}\ R_{\lambda\lambda}^{(V,V)} \right) \right.$$

$$\left. + |G_1|^2 \left(2R_{\mu\lambda}^{(A,A)}\ I_{\mu\lambda} - I_{\omega\omega} R_{\lambda\lambda}^{(A,A)} \right) \right]$$

Using Table 15 and equation 19 gives

$$I_{\omega\omega}\ R_{\lambda\lambda}^{(V,V)} - 2R_{\mu\lambda}^{(V,V)}\ I_{\mu\lambda} = \tag{49}$$

$$= 16m_1m_2(4\alpha + \beta Q^2) + 8q_1q_2(4\alpha + \beta Q^2)$$

$$-\ 32\alpha(m_1m_2 + q_1q_2) - 8\beta Q^2(m_1m_2 + q_1q_2)$$

$$+\ 16\alpha q_1q_2 + 16\beta(q_1Q)(q_2Q)$$

$$= 8m_1m_2(4\alpha + \beta Q^2) + 16\ \alpha q_1q_2 + 16\beta(q_1Q)(q_2Q)$$

Similarly

$$-I_{\omega\omega}\ R_{\lambda\lambda}^{(A,A)} + 2R_{\mu\lambda}^{(A,A)}\ I_{\mu\lambda} \tag{50}$$

$$= -8m_1m_2(4\alpha+\beta Q^2) + 16\alpha q_1q_2 + 16\beta(q_1Q)(q_2Q)$$

Now since $Q = q_1 - q_2$ it follows that

$$Q^2 = -m_1^2 - m_2^2 - 2q_1 q_2$$

or

$$q_1 q_2 = -1/2(Q^2 + m_1^2 + m_2^2) \tag{51}$$

Similarly

$$q_1 \ Q = 1/2(Q^2 + m_2^2 - m_1^2) \tag{52}$$

and

$$q_2 Q = -1/2(Q^2 + m_1^2 - m_2^2) \tag{53}$$

Thus it is clear that the right sides of equations 49 and 50 can be expressed in terms of the single variable Q^2. Substituting from equations 51 - 53 gives

$$I_{\omega\omega} \ R_{\lambda\lambda}^{(V,V)} - 2R_{\mu\lambda}^{(V,V)} \ I_{\mu\lambda} \tag{54}$$

$$= 8m_1 m_2(4\alpha + \beta Q^2) + 16\alpha(-1/2)(Q^2 + m_1^2 + m_2^2)$$

$$+ 16\beta(-1/4)(Q^2 + m_2^2 - m_1^2)(Q^2 + m_1^2 - m_2^2)$$

$$= \frac{\pi}{3} R_V(-Q^2)\Theta(-Q^2 - m_e^2)\Theta(Q)$$

where the function R_V has been introduced for convenience:

$$R_V(-Q^2) = \frac{3}{\pi\Theta(-Q^2 - m_e^2)\Theta(Q)} \Big[-4Q^2(2\alpha + \beta Q^2) + 8m_1 m_2(4\alpha + \beta Q^2) \qquad (55)$$

$$- 8\alpha(m_1^2 + m_2^2) + 4\beta(m_1^2 - m_2^2)^2 \Big]$$

In a similar fashion it follows that

$$-I_{\omega\omega} R_{\lambda\lambda}^{(A,A)} + 2R_{\mu\lambda}^{(A,A)} I_{\mu\lambda} = \frac{\pi}{3} R_A(-Q^2)\Theta(-Q^2 - m_e^2)\Theta(Q) \qquad (56)$$

where $R_A(-Q^2)$ differs from $R_V(-Q^2)$ only by the sign on the term involving the product $m_1 m_2$. This is easily seen by comparing equations 49 and 50. In terms of R_V and R_A the partial decay width (equation 48) becomes

$$\Gamma(b_1 \rightarrow b_2\, e^-\, \bar{\nu}) = \frac{g^2}{96\,\pi^4 E_1} \int dq_2\, \delta(q_2^2 + m_2^2)\Theta(q_2)\Theta(-Q^2 - m_e^2)\Theta(Q)$$

$$\times \Big[|F_1|^2 R_V(-Q^2) + |G_1|^2 R_A(-Q^2) \Big]$$

$$= \frac{g^2}{96\,\pi^4 E_1} \int_{m_e^2}^{\infty} du \Big[|F_1|^2 R_V(u) + |G_1|^2 R_A(u) \Big]$$

$$\times \int dq_2\, \delta(q_2^2 + m_2^2)\Theta(q_2)\Theta(q_1 - q_2)\delta\Big((q_1 - q_2)^2 + u \Big)$$

$$(57)$$

The last equality is readily verified by performing the u-integration with the aid of the last δ-function. The lower limit of the u-integration is determined by the function $\Theta(-Q^2 - m_e^2)$.

Final Baryon Momentum Integration

Let I denote the integral on q_2 in the last line of equation 57, then I can be written in the form:

$$I = \int d^3q_2 \int dq_o \ \delta(\vec{q}_2^{\,2} - q_{2o}^2 + m_2^2)\delta(-m_1^2 - m_2^2 - 2q_1q_2 + u) \tag{58}$$

$$\times \ \Theta(q_2)\Theta(q_1 - q_2)$$

From its form in equation 57 it is clear that I is a Lorentz invariant. Hence it can be evaluated without loss of generality in the rest frame of b_1 where $\vec{q}_1 = 0$ and $E_1 = m_1$. In this frame

$$I = \int d^3q_2 \int \frac{dq_{2o}}{2q_{2o}} \delta \left(q_{2o} - \sqrt{\vec{q}_2^{\,2} + m_2^2} \right) \tag{59}$$

$$\times \ \delta(-m_1^2 - m_2^2 + 2m_1q_{2o} + u) \ \Theta(q_1 - q_2)$$

$$= 2\pi \int \frac{|\vec{q}_2|^2 d|\vec{q}_2|}{E_2} \delta(-m_1^2 - m_2^2 + 2m_1E_1E_2 + u)\Theta(m_1 - E_2)$$

where $E_2 = \sqrt{\vec{q}_2^{\,2} + m_2^2}$

This integration can be performed by making a change of variables:

$$x = 2m_1E_2 = 2m_1\sqrt{\vec{q}_2^{\,2} + m_2^2} \tag{60}$$

$$dx = \frac{2m_1|\vec{q}_2|d|\vec{q}_2|}{\sqrt{\vec{q}_2^{\,2} + m_2^2}} = \frac{2m_1|\vec{q}_2|d|\vec{q}_2|}{E_2}$$

Thus

$$I = 2\pi \int \frac{|\vec{q}_2|}{2m_1} \; dx \; \delta(-m_1^2 - m_2^2 + x + u)\Theta(m_1 - x/2m_1)\Theta(x - 2m_1m_2)$$

$$= \frac{\pi}{m_1} \left. |\vec{q}_2|\Theta(m_1 - x/2m_1)\Theta(x - 2m_1m_2) \right|_{x=m_1^2+m_2^2-u} \qquad (61)$$

The second Θ-function in equation 61 simply expresses the lower limit of integration on x as determined by equation 60. Also from equation 60

$$\left. \sqrt{\vec{q}_2^{\,2} + m_2^2} \right|_{x=m_1^2+m_2^2-u} = \frac{m_1^2 + m_2^2 - u}{2m_1}$$

so that

$$\vec{q}_2^{\,2} = \frac{m_1^4 + m_2^4 + u^2 + 2m_1^2\,m_2^2 - 2m_1^2\,u - 2m_2^2\,u}{4m_1^2} - m_2^2 \qquad (62)$$

$$= \frac{\lambda(u,\, m_1^2, m_2^2)}{4m_1^2}$$

where λ is the quadratic form defined by

$$\lambda(x,\, y,\, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz \qquad (63)$$

Similarly the arguments of the Θ-functions are

$$\left. m_1 - \frac{x}{2m_1} \right|_{x=m_1^2+m_2^2-u} = m_1 - \frac{m_1^2 + m_2^2 - u}{2m_1} \qquad (64)$$

$$= \frac{m_1^2 - m_2^2 + u}{2m_1}$$

and

$$x - 2m_1m_2\Big|_{x=m_1^2+m_2^2-u} = m_1^2 + m_2^2 - u - 2m_1m_2 \tag{65}$$

$$= (m_1 - m_2)^2 - u$$

Hence

$$I = \frac{\pi\sqrt{\lambda(u,m_1^2,\ m_2^2)}}{2m_1^2}\ \Theta(u + m_1^2 - m_2^2)\Theta((m_1 - m_2)^2 - u) \tag{66}$$

The partial decay width (equation 57) can now be written in the rest frame of b_1 as

$$\Gamma(b_1 \longrightarrow b_2\ e^-\ \bar{\nu}) = \frac{g^2}{192\pi^3 m_1^3}\int_{m_e^2}^{(m_1-m_2)^2} \sqrt{\lambda(u,m_1^2,m_2^2)}\ \Big[|F_1|^2\ R_V(u) + |G_1|^2 R_A(u)\Big]\ du \tag{67}$$

the upper limit of the u-integration was determined by the second Θ-function in equation 66 while the first Θ-function is superseded by the lower limit m_e^2.

Integration on the Square of the Momentum Transfer

The remaining integration in equation 67 can be performed explicitly. However, the result is rather complicated and not very informative. There are two limiting cases where the result is relatively simple. Case 1: The energy available for the decay is much less than the mass of the original baryon, i.e., $m_1 - m_2 < m_1$. In this case both the electron mass and the mass difference $m_1 - m_2$ can be neglected when compared with the original baryon mass; however, they can not be neglected when compared to one another. Neutron decay is an example of this situation. One then has the familiar result (see ref. 71)

$$\Gamma(b_1 \rightarrow b_2\ e^-\ \bar{\nu}) = \frac{g^2 m_e^5}{2\pi^3} \left(|F_1|^2 + 3|G_1|^2\right) f(\eta) \tag{68}$$

where

$$f(\eta) = \frac{1}{30}\eta^5 - \frac{1}{12}\eta^3 - \frac{1}{4}\eta + \frac{1}{4}\sqrt{\eta^2 + 1}\ \ln\left(\eta + \sqrt{\eta^2 + 1}\right)$$

and

$$\eta = \frac{\sqrt{(m_1 - m_2)^2 - m_e^2}}{m_e} .$$

Case 2: The energy available for the decay is not negligible when compared with the original baryon mass. In this case the electron mass can be neglected when compared with the original baryon mass and when compared with the mass difference $m_1 - m_2$. It is this situation which will be considered in the remainder of this section. Hence in equation 67 it is permissible to set $m_e = 0$.

In order to establish the explicit form of the integrand it is necessary to refer to equations 34, 37, 38, 55 and 56. For $m_e = 0$ equations 37 and 38 reduce to

$$\beta = \frac{2}{3Q^2} \quad \varphi = \frac{\pi}{12}\,\Theta(-Q^2)\Theta(Q) \tag{69}$$

and

$$\alpha = 1/3\ \varphi = \frac{\pi}{24} Q^2 \Theta(-Q^2)\Theta(Q) \tag{70}$$

respectively. Equation 55 then becomes

$$R_V(-Q^2) = \frac{1}{4}\Big[-4Q^2(2Q^2) + 8m_1m_2(3Q^2) - 4Q^2(m_1^2 + m_2^2) + 4(m_1^2 - m_2^2)^2$$

or

$$R_V(u) = -2u^2 + (m_1^2 + m_2^2 - 6m_1m_2)u + (m_1^2 - m_2^2)^2 \tag{71}$$

From the discussion following equation 56 it follows that

$$R_A(u) = -2u^2 + (m_1^2 + m_2^2 + 6m_1m_2)u + (m_1^2 - m_2^2)^2 \tag{72}$$

It is convenient at this point to make a change of variables. Let $z = \frac{u}{\Delta^2}$ where $\Delta = m_1 - m_2$. At the same time setting $\xi = \frac{\Delta}{m_1}$ it follows that $m_1 = \frac{\Delta}{\xi}$ and $m_2 = m_1 - \Delta = \frac{\Delta}{\xi}(1 - \xi)$. Thus

$$\lambda(u, m_1^2, m_2^2) = \Delta^4(z^2 + az + b) \tag{73}$$

where

$$a = \frac{-2}{\xi^2}(2 - 2\xi + \xi^2) \tag{74}$$

and

$$b = \left(\frac{2 - \xi}{\xi}\right)^2 \tag{75}$$

Similarly

$$R_V(u) = \Delta^4(-2z^2 + c_V z + b) \tag{76}$$

where

$$c_V = \frac{1}{\xi^2}(-4 + 4\xi + \xi^2) \tag{77}$$

Also

$$R_A(u) = \Delta^4(-2z^2 + c_A z + b) \tag{78}$$

where

$$c_A = \frac{1}{\xi^2}(8 - 8\xi + \xi^2) \tag{79}$$

Making these substitutions in equation 67 gives

$$\Gamma(b_1 \rightarrow b_2\ e^-\ \bar{\nu}) = \frac{g^2\xi^3\Delta^5}{192\ \pi^3} \int_0^1 dz \sqrt{z^2 + az + b} \tag{80}$$

$$\times \left[|F_1|^2(-2z^2 + c_V z + b) + |G_1|^2(-2z^2 + c_A z + b)\right]$$

The remaining integrals can be evaluated in a straight forward, though tedious, fashion. The details are given in Appendix 2. The results are

$$\int_0^1 (-2z^2 + c_V z + b)\sqrt{z^2 + az + b}\, dz \tag{81}$$

$$= \frac{16}{5\xi^3}\left(1 - 3/2\xi + 6/7\xi^2 - 5/28\xi^3 - 1/112\xi^4 + \ldots\right)$$

and

$$\int_0^1 (-2z^2 + c_A z + b)\sqrt{z^2 + az + b}\, dz = \tag{82}$$

$$= \frac{48}{5\xi^3}\left(1 - 3/2\xi + 4/7\xi^2 - 1/28\xi^3 + 1/144\xi^4 + \ldots\right)$$

Substituting in equation 81 gives

$$\Gamma(b_1 \longrightarrow b_2 e^- \bar{\nu}) = \frac{g^2\Delta^5}{60\pi^3}\left[|F_1|^2 f_V(\xi) + 3|G_1|^2 f_A(\xi)\right] \tag{83}$$

where

$$f_V(\xi) = 1 - 3/2\xi + 6/7\xi^2 - 5/28\xi^3 - 1/112\xi^4 + \ldots$$

and

$$f_A(\xi) = 1 - 3/2\xi + 4/7\xi^2 - 1/28\xi^3 + 1/144\xi^4 + \ldots$$

Numerical Results

Equation 83 can now be used to evaluate the branching ratios $\dfrac{\Gamma(b_1 \rightarrow b_2 e^- \bar{\nu})}{\Gamma(b_1 \rightarrow \text{all})}$ in all the theories considered previously: the U.F.I., the Cabibbo theory and the schizon model of lepton currents. Only three decays will be considered since these will be sufficient for drawing at least preliminary conclusions about the schizon model. The decays to be considered are those listed previously in Table 4. The values of F_1 and G_1 for these decays in the different theories are listed in Table 16. For the schizon theory axial vector renormalization is assumed to be the same as in the U.F.T. The numerical values were obtained either directly from references 19 and 28 or were evaluated using the numbers given in these references.

Using Table 16 along with equation 83 the branching ratios of the three decays were evaluated and the results are tabulated in Table 17.

Table 16. Values for F_1 and G_1

Decay		U.F.I.	Cabibbo	Schizon
$\Lambda \rightarrow pe^- \bar{\nu}$	F_1	1	$-\sqrt{3/2}\ \sin\Theta$	$-1/\sqrt{6}$
	G_1	x	$-1/2\ \sqrt{3/2}\ \sin\Theta(H^E + H^O)$	$-x/\sqrt{6}$
$\Sigma^- \rightarrow ne^- \bar{\nu}$	F_1	1	$\sin\Theta$	1
	G_1	x	$-1/2\ \sin\Theta(H^E - H^O)$	x
$\Sigma^- \rightarrow \Lambda e^- \bar{\nu}$	F_1	1	0	$2/\sqrt{6}$
	G_1	x	$1/2\ \sqrt{2/3}\ \cos\Theta(H^E)$	$2x/\sqrt{6}$

$x = 1.18 \pm 0.02$ $\sin\Theta = 0.206$ $1/2\ H^E = 0.81 \pm 0.14$

$\cos\Theta = 0.978 \pm 0.003$ $1/2\ H^O = 0.37 \pm 0.16$

Table 17. Comparison of Results

Decay	Experiment	Branching Ratios ($X10^3$) U.F.I.	Cabibbo	Schizon
$\Lambda \rightarrow pe^- \bar{\nu}$	0.85 ± 0.09	19.1 ± 3.1	$0.55 {+0.31 \atop -0.21}$	3.19 ± 0.51
$\Sigma^- \rightarrow ne^- \bar{\nu}$	1.3 ± 0.2	70.6 ± 5.5	$0.99 {+0.72 \atop -0.35}$	70.6 ± 5.5
$\Sigma^- \rightarrow \Lambda e^- \bar{\nu}$	0.074 ± 0.02	0.296 ± 0.012	0.074 ± 0.02*	0.197 ± 0.008

*Input information.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

It is clear from Table 17 that the schizon model is at least as good as the U.F.I. and in two of the three cases considered is much better than the U.F.I. However neither of these two theories compare favorably with the Cabibbo theory which appears to be within experimental error in all instances. It should be kept in mind that the last entry of Table 17 is input information for the Cabibbo theory. This input information is not needed in the schizon model.

The fact that the schizon model based upon pure octet P_{13} invariance is an improvement over the U.F.I. and that it predicts relative strengths of strangeness conserving and strangeness non-conserving processes encourages one to attempt modifications which would bring its predictions into agreement with the experimental results. The simplest such modification would be to drop the third assumption of the model. This would permit the use of admixtures of the other three P_{24} invariants with eigenvalues -1.

There are other possibilities which could be considered as well. Five such possibilities, all of which are outside the schizon model but retain P_{24} invariance as a basic constituent, are listed below.

1. Find a reasonable lepton octet. If such an octet exists then the weak interaction may be describable by a P_{24} invariant biquadratic SU(3) scalar (perhaps even the pure octect scalar).

2. Consider similar arguments in an SU(6) symmetry scheme (72).
3. Consider a quark (73) or other fundamental field model (74). The feeling here is that perhaps weak interactions are governed not by the baryons themselves but by their underlying fields. The underlying fields then might partake in weak interactions in a P_{24} invariant fashion.
4. Consider the recently discussed U(4) symmetry schemes (75) for weak interactions. These are particularly appealing since they allow a rather nice baryon-lepton analogy.
5. Assume that the weak interaction is not P_{24} invariant but has a certain P_{24} property which reduces to P_{24} invariance when only nucleons, electrons and neutrinos are involved.

In the opinion of the author the last three possibilities are worthy of further investigation.

APPENDIX 1

THE DIRAC EQUATION AND SPIN SUMS

Free particles with mass m and spin 1/2 are described by the Dirac equation*

$$(\gamma_\mu \partial_\mu + m)\,\psi(x) = 0\ , \qquad \partial_\mu = (\partial_x, \partial_y, \partial_z, -i\partial_t) \tag{1}$$

where the Dirac matrices γ_μ satisfy

$$\left\{\gamma_\mu, \gamma_\nu\right\} = 2\delta_{\mu\nu} \tag{2}$$

and

$$\gamma_\mu = \gamma_\mu^\dagger \tag{3}$$

A convenient representation for the matrices γ_μ is the following

$$\gamma_k = -i\beta\alpha_k\ , \qquad k = 1, 2, 3;\ \ \gamma_4 = \beta \tag{4}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \tag{5}$$

where σ is the 2 x 2 Pauli matrix vector. In this representation

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

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*Greek subscripts take on the values 1, 2, 3, 4 while Roman subscripts take on the values 1, 2, 3. The summation convention on repeated indices is used throughout.

Taking the adjoint of the Dirac equation gives

$$\psi^{\dagger}(x)(\gamma_k \partial_k - \gamma_4 \partial_4 + m) = 0 \tag{7}$$

Multiplying on the right by γ_4 and commuting gives

$$\overline{\psi}(x)(\gamma_\mu \partial_\mu - m) = 0 \tag{8}$$

where

$$\overline{\psi}(x) = \psi^{\dagger}\gamma_4$$

Consider now the plane wave solutions of the Dirac equation

$$\psi(x) = \frac{1}{\sqrt{V}} \; e^{iqx} \; u(\vec{q}) \tag{9}$$

Here $qx = \vec{q} \cdot \vec{x} - q_o t$ where $q = (\vec{q}, \, iq_o)$ and $x = (\vec{x}, \, it)$. Substituting in the Dirac equation gives

$$(i\gamma_k q_k - \gamma_4 q_o + m)u(\vec{q}) = 0 \tag{10}$$

Then letting $u = \begin{pmatrix} v \\ w \end{pmatrix}$ gives

$$\begin{pmatrix} m - q_o & \vec{\sigma} \cdot \vec{q} \\ -\vec{\sigma} \cdot \vec{q} & m + q_o \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0 \tag{11}$$

or

$$w = \frac{\vec{\sigma} \cdot \vec{q}}{m + q_o} \, v \tag{12}$$

and

$$v = \frac{-\vec{\sigma} \cdot \vec{q}}{m - q_o} w \tag{13}$$

this requires that

$$w = \left(\frac{\vec{\sigma} \cdot \vec{q}}{m + q_o}\right)\left(\frac{\vec{\sigma} \cdot \vec{q}}{q_o - m}\right) w = \frac{\vec{q}^{\,2}}{q_o^2 - m^2} \; w \tag{14}$$

or

$$q_o^2 = \vec{q}^{\,2} + m^2$$

Thus

$$q_o = \pm\sqrt{\vec{q}^{\,2} + m^2} \tag{15}$$

In the following, as in Chapter V,

$$E = |q_o| = \sqrt{\vec{q}^{\,2} + m^2} \tag{16}$$

and

$$q = (\vec{q}, iE)$$

The renormalized solutions for u can now be written down.

$$u^{(+)(r)}(q) = \sqrt{\frac{E + m}{2m}} \begin{pmatrix} \xi^{(r)} \\ \dfrac{\vec{\sigma} \cdot \vec{q}}{E + m} \; \xi^{(r)} \end{pmatrix}$$

$$u^{(-)(r)}(q) = (-1)^s \sqrt{\frac{E + m}{2m}} \begin{pmatrix} \dfrac{-\vec{\sigma} \cdot \vec{q}}{E + m} \; \xi^{(s)} \\ \xi^{(s)} \end{pmatrix}, \quad r \neq s \tag{17}$$

In equations 17 the $+$ and $-$ denote respectively the positive and negative energy solutions, r denotes the spin and

$$\xi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$$
$$\xi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix} \tag{18}$$

The solutions in equation 17 form a complete orthonormal set for fixed momentum $\vec{q}$:

$$u^{(+)(r)\dagger}(\vec{q})u^{(+)(s)}(\vec{q}) = u^{(-)(r)\dagger}(\vec{q})u^{(-)(s)}(\vec{q}) = \delta_{rs}$$
$$u^{(+)(r)\dagger}(\vec{q})u^{(-)(s)}(\vec{q}) = 0 \tag{19}$$

$$\sum_{r=1}^{2}\left[u^{(+)(r)}(\vec{q})u^{(+)(r)\dagger}(\vec{q}) + u^{(-)(r)}(\vec{q})u^{(-)(r)\dagger}(\vec{q})\right] = 1 \tag{20}$$

Furthermore these solutions satisfy the following equation

$$(i\vec{\gamma}\cdot\vec{q} - \gamma_4\eta E + m)u^{(\eta)(r)}(\vec{q}) = 0\ , \qquad \eta = \pm 1 \tag{21}$$

As can be seen from equation 10. Replacing $\vec{q}$ by $\eta\vec{q}$ in this equation gives

$$(i\eta\gamma q + m)u^{(\eta)(r)}(\eta\vec{q}) = 0 \tag{22}$$

Now multiplying the completeness relations (equation 20) by γ_4 on the right and by $\gamma_4(i\gamma q + m)$ on the left and using equation 22 gives

$$\sum_{r=1}^{2} \gamma_4(i\gamma q + m)u^{(-)(r)}(\vec{q})u^{(-)(r)}(\vec{q}) = \gamma_4(i\gamma q + m)\gamma_4 \qquad (23)$$

Now replacing $\vec{q}$ by $-\vec{q}$ and commuting the γ_4's gives

$$\gamma_4(i\gamma q + m) \longrightarrow \gamma_4(-i\vec{\gamma}\cdot\vec{q} - \gamma_4 E + m) \qquad (24)$$

$$= \gamma_4(-i\vec{\gamma}\cdot\vec{q} + \gamma_4 E + m - 2\gamma_4 E)$$

$$= \gamma_4(-i\gamma q + m) - 2E$$

and

$$\gamma_4(i\gamma q + m)\,\gamma_4 \longrightarrow \gamma_4(-i\vec{\gamma}\cdot\vec{q} - \gamma_4 E + m)\gamma_4 \qquad (25)$$

$$= i\vec{\gamma}\cdot\vec{q} - \gamma_4 E + m$$

$$= i\gamma q + m$$

Substituting in equation 23 and using equation 22 again gives

$$\sum_{r=1}^{2} u^{(-)(r)}(-\vec{q})\bar{u}^{(-)(r)}(-\vec{q}) = -\frac{1}{2E}(i\gamma q + m) \qquad (26)$$

Similarly

$$\sum_{r=1}^{2} u^{(+)(r)}(\vec{q})\bar{u}^{(+)(r)}(\vec{q}) = \frac{-1}{2E}(i\gamma q - m) \qquad (27)$$

These expressions for evaluating spin sums can be combined into a single convenient expression

$$\sum_{r=1}^{2} u^{(\eta)(r)}(\eta\vec{q})\bar{u}^{(\eta)(r)}(\eta\vec{q}) = -\frac{1}{2E}(i\gamma q - \eta m) \tag{28}$$

APPENDIX 2

SOME INTEGRALS APPEARING IN THE BARYON BETA DECAYS

The integrals appearing in equation 80 of Chapter V can be evaluated in a straight forward fashion. These integrals will be evaluated in this appendix using equations 181, 187, 198 and 201 of the Table of Integrals of reference 76. The results will be expanded in powers of ξ, defined after equation 72 in Chapter V.

$$\int_0^1 dz\sqrt{z^2 + az + b} = \frac{(a + 2)\sqrt{a + b + 1} - a\sqrt{b}}{4} + \frac{4b - a^2}{8}\int_0^1 \frac{dz}{\sqrt{z^2 + az + b}} \tag{1}$$

$$= \frac{(a + 2)\sqrt{a + b + 1} - a\sqrt{b}}{4} + \frac{4b - a^2}{8}\ln\frac{\sqrt{a + b + 1} + 1 + a/2}{\sqrt{b} + 1}$$

Using equations 74 and 75 of Chapter V gives

$$a + 2 = \frac{4(\xi - 1)}{\xi^2} \tag{2}$$

$$a + b + 1 = 0 \tag{3}$$

$$a - 2\sqrt{b} = -4/\xi^2 \tag{4}$$

and

$$a + 2\sqrt{b} = -\frac{4}{\xi^2}(\xi - 1)^2 \tag{5}$$

Thus

$$\int_0^1 dz\sqrt{z^2 + az + b} = \frac{(2-\xi)(2 - 2\xi + \xi^2)}{2\xi^3} + \frac{2(1-\xi)^2}{\xi^4}\ln(1-\xi) \tag{6}$$

It is convenient at this point to expand in powers of ξ . Using

$$\ln(1-\xi) = -\xi(1 + 1/2\xi + 1/3\xi^2 + 1/4\xi^3 + 1/5\xi^4 + 1/6\xi^5 + 1/7\xi^6 + 1/8\xi^7 + 1/9\xi^8 + \ldots)$$

and performing the indicated multiplications gives

$$\int_0^1 dz\sqrt{z^2 + az + b} = \frac{1}{\xi}(4/3 - 2/3\xi - 1/15\xi^2 - 1/30\xi^3 - 2/105\xi^4 - 1/84\xi^5 - 1/126\xi^6 + \ldots) \tag{7}$$

Similarly

$$\int_0^1 dz\, z\sqrt{z^2 + az + b} = \frac{(a+b+1)\sqrt{a+b+1}}{3} - \frac{b\sqrt{b}}{3} - \frac{a}{2}\int_0^1 dz\sqrt{z^2 + az + b} \tag{8}$$

Now

$$\frac{-b\sqrt{b}}{3} = \frac{1}{\xi^3}\,(-8/3 + 4\xi - 2\xi^2 + 1/3\xi^3)$$

and

$$\frac{-a}{2}\int_0^1 dz\sqrt{z^2 + az + b} = \frac{1}{\xi^5}\,(2 - 2\xi + \xi^2)(4/3\xi^2 - 2/3\xi^3$$

$$- 1/15\xi^4 - 1/30\xi^5 - 2/105\xi^6 - 1/84\xi^7 - 1/126\xi^8 + \ldots)$$

Thus, substituting in equation 8 gives

$$\int_0^1 dz\, z\sqrt{z^2 + az + b} = \frac{1}{\xi^3}\,(8/15\xi^2 - 4/15\xi^3 - 4/105\xi^4 - \tag{9}$$

$$-2/105\xi^5 - 1/90\xi^6 + \ldots)$$

In a similar fashion

$$\int_0^1 dz\, z^2\sqrt{z^2 + az + b} = \left(1 - \frac{5a}{6}\right)\frac{(a + b + 1)\sqrt{a + b + 1}}{4} \tag{10}$$

$$+ \frac{5ab\sqrt{b}}{24} + \frac{5a^2 - 4b}{16}\int_0^1 dz\sqrt{z^2 + az + b}$$

Now

$$\frac{5a^2 - 4b}{16} = \frac{1}{\xi^4}\,(5 - 10\xi + 9\xi^2 - 4\xi^3 + \xi^4)$$

Hence

$$\frac{5a^2 - 4b}{16} \int_0^1 dz \sqrt{z^2 + az + b} = \frac{1}{\xi^5} (5 - 10\xi + 9\xi^2 - 4\xi^3 + \xi^4)$$

$$\times (4/3 - 2/3\xi - 1/15\xi^2 - 1/30\xi^3 - 2/105\xi^4$$

$$- 1/84\xi^5 - 1/126\xi^6 + \ldots)$$

Also

$$\frac{5ab\sqrt{b}}{24} = \frac{1}{\xi^5} (-20/3 + 50/3\xi - 55/3\xi^2 + 65/6\xi^3 - 10/3\xi^4 + 5/12\xi^5)$$

Thus, substituting in equation 10 gives

$$\int_0^1 dz\ z^2\sqrt{z^2 + az + b} = \frac{1}{\xi^3} (32/105\xi^2 - 16/105\xi^3 - 8/315\xi^4 + \ldots) \tag{11}$$

Now using equations 7, 9, 11, V-75, V-77 and V-79 gives

$$\int_0^1 dz(-2z^2 + c_V z + b) \sqrt{z^2 + az + b} = \tag{12}$$

$$= \frac{16}{5\xi^3} (1 - 3/2\xi + 6/7\xi^2 - 5/28\xi^3 - 1/112\xi^4 + \ldots)$$

and

$$\int_0^1 dz(-2z^2 + c_A z + b) \sqrt{z^2 + az + b} = \tag{13}$$

$$= \frac{48}{5\xi^3} (1 - 3/2\xi + 4/7\xi^2 - 1/28\xi^3 + 1/144\xi^4 + \ldots)$$

Equations 12 and 13 are the integrals required for the evaluation of equation V-80.

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